Show That The Quadrilateral Having Vertices At (1, 2, 3), (4,3, 1), (2, 2, 1) And (5, 7, 3) Is A Parallelogram,

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Understanding whether a quadrilateral is a parallelogram is a fundamental problem in coordinate geometry. When given the vertices of a quadrilateral in three-dimensional space, we can utilize vector methods to verify its properties. In this article, we will explore the process of determining if the quadrilateral with vertices at (1, 2, 3), (4, 3, 1), (2, 2, 1), and (5, 7, 3) is a parallelogram. We will discuss the necessary background, step-by-step calculations, and logical reasoning to arrive at the conclusion.

Understanding the Problem and Key Concepts

Before delving into calculations, it's essential to understand the core concepts involved in identifying a parallelogram in 3D space.

What Is a Parallelogram?

A parallelogram is a quadrilateral with both pairs of opposite sides parallel. In coordinate geometry, this property translates into specific vector relationships among the vertices.

Key Properties of Parallelograms in Coordinates

- Opposite sides are equal and parallel: The vectors representing opposite sides are equal.
- Diagonals bisect each other: The midpoints of the diagonals coincide.

Approach to the Problem

To verify whether the given quadrilateral is a parallelogram, we can:
- Check if the vectors representing opposite sides are equal.
- Or, verify if the diagonals bisect each other by comparing their midpoints.

Using these properties, our calculations will focus on vector operations and midpoint formulas.

Step 1: Labeling the Vertices

Let's denote the vertices as follows:
- $A = (1, 2, 3)$
- $B = (4, 3, 1)$
- $C = (2, 2, 1)$
- $D = (5, 7, 3)$

The order of vertices suggests the quadrilateral is formed as $A \to B \to C \to D \to A$, but the specific order may impact the calculations. Our goal is to verify if this arrangement forms a parallelogram.

Step 2: Calculating Vectors for Opposite Sides

To check if the quadrilateral is a parallelogram, we examine vectors corresponding to pairs of opposite sides.

Vectors for adjacent sides:

- Side AB: $\vec{AB} = B - A = (4 - 1, 3 - 2, 1 - 3) = (3, 1, -2)$
- Side BC: $\vec{BC} = C - B = (2 - 4, 2 - 3, 1 - 1) = (-2, -1, 0)$
- Side CD: $\vec{CD} = D - C = (5 - 2, 7 - 2, 3 - 1) = (3, 5, 2)$
- Side DA: $\vec{DA} = A - D = (1 - 5, 2 - 7, 3 - 3) = (-4, -5, 0)$

Vectors for opposite sides:

- Opposite sides AB and CD:
- $\vec{AB} = (3, 1, -2)$
- $\vec{CD} = (3, 5, 2)$

- Opposite sides BC and DA:
- $\vec{BC} = (-2, -1, 0)$
- $\vec{DA} = (-4, -5, 0)$

Observation:
If the quadrilateral is a parallelogram, then:
- $\vec{AB}$ should be equal to $\vec{DC}$ (or $\vec{CD}$ equals $\vec{AB}$), and
- $\vec{BC}$ should be equal to $\vec{DA}$.

Since $\vec{AB} \neq \vec{CD}$ and $\vec{BC} \neq \vec{DA}$, these sides are not directly equal. But, perhaps the order of vertices is different, or the quadrilateral is not necessarily in the order $A \to B \to C \to D$.

Step 3: Checking Diagonals for Midpoint Coincidence

Since the opposite sides are not matching in the first check, an alternative is to verify if the diagonals bisect each other, which is a hallmark of parallelograms.

Calculating Diagonals:

- Diagonal AC: $\vec{AC} = C - A = (2 - 1, 2 - 2, 1 - 3) = (1, 0, -2)$
- Diagonal BD: $\vec{BD} = D - B = (5 - 4, 7 - 3, 3 - 1) = (1, 4, 2)$

Midpoints of the diagonals:

- Midpoint of AC:

$$
M_{AC} = \frac{A + C}{2} = \left(\frac{1 + 2}{2}, \frac{2 + 2}{2}, \frac{3 + 1}{2} \right) = (1.5, 2, 2)
$$

- Midpoint of BD:

$$
M_{BD} = \frac{B + D}{2} = \left(\frac{4 + 5}{2}, \frac{3 + 7}{2}, \frac{1 + 3}{2} \right) = (4.5, 5, 2)
$$

Comparison:

Since $M_{AC} \neq M_{BD}$, the diagonals do not bisect each other at the same point. Therefore, the quadrilateral is not a parallelogram based solely on this criterion.

Step 4: Re-examining the Vertex Arrangement

Given the previous results, perhaps the vertices are ordered differently or the shape is not a typical quadrilateral in the sequence assumed. Let's check the other possible pairings:

- Diagonals $A \to D$ and $B \to C$:

Calculating:

- Diagonal AD: $D - A = (5 - 1, 7 - 2, 3 - 3) = (4, 5, 0)$
- Diagonal BC: $C - B = (2 - 4, 2 - 3, 1 - 1) = (-2, -1, 0)$

Midpoints:

- Midpoint of AD:
$$
M_{AD} = \left(\frac{1 + 5}{2}, \frac{2 + 7}{2}, \frac{3 + 3}{2} \right) = (3, 4.5, 3)
$$
- Midpoint of BC:
$$
M_{BC} = \left(\frac{4 + 2}{2}, \frac{3 + 2}{2}, \frac{1 + 1}{2} \right) = (3, 2.5, 1)
$$

Again, the midpoints do not coincide, indicating that the diagonals do not bisect at the same point, thus not forming a parallelogram.

Step 5: Geometric Interpretation and Final Conclusion

Based on the calculations:
- The vectors for opposite sides are not equal.
- The diagonals do not bisect each other at the same midpoint.

Therefore, the quadrilateral with vertices at (1, 2, 3), (4, 3, 1), (2, 2, 1), and (5, 7, 3) does not satisfy the key properties of a parallelogram in three-dimensional space.

Summary of Findings

- The vector analysis of sides indicates that the pairs of opposite sides are not equal and parallel.
- The midpoints of diagonals do not coincide, ruling out the possibility of the shape being a parallelogram.
- The shape formed by these vertices is likely a general quadrilateral, not a parallelogram.

Conclusion

After thorough coordinate and vector computations, it is evident that the quadrilateral with vertices at (1, 2, 3), (4, 3, 1), (2, 2, 1), and (5, 7, 3) is not a parallelogram. This conclusion is based on the fundamental properties of parallelograms—equal

Frequently Asked Questions

How can we prove that a quadrilateral with vertices at (1, 2, 3), (4, 3, 1), (2, 2, 1), and (5, 7, 3) is a parallelogram?

To prove the quadrilateral is a parallelogram, we can show that its opposite sides are equal and parallel by calculating the vectors of the sides and verifying that their pairs are equal or parallel. Alternatively, we can verify that the diagonals bisect each other by checking if their midpoints coincide.

What steps are involved in verifying whether the given vertices form a parallelogram in 3D space?

The steps include: 1) Find the vectors representing the sides or diagonals; 2) Check if opposite sides are equal and parallel by comparing vectors; 3) Or, compute the midpoints of the diagonals and verify if they are the same, indicating the diagonals bisect each other, which is a property of parallelograms.

Can the midpoint formula be used to confirm that the quadrilateral with given vertices is a parallelogram?

Yes. By calculating the midpoints of both diagonals and verifying they are the same point, we confirm that the diagonals bisect each other, which is a defining property of a parallelogram.

What is the significance of vector addition in demonstrating that a quadrilateral is a parallelogram?

Vector addition helps by showing that opposite sides are equal and parallel. If the vectors representing opposite sides are equal, then the quadrilateral is a parallelogram. Specifically, if AB + CD = 0 or the vectors AB and CD are equal and parallel, it confirms the parallelogram property.

How do you calculate the midpoints of diagonals for the vertices (1, 2, 3), (4, 3, 1), (2, 2, 1), and (5, 7, 3) to verify if it's a parallelogram?

Calculate the midpoints of diagonals AC and BD using the midpoint formula: Midpoint = ((x1 + x2)/2, (y1 + y2)/2, (z1 + z2)/2). If these midpoints are the same, it confirms that the diagonals bisect each other, proving the quadrilateral is a parallelogram.

Additional Resources

Parallelogram: An In-Depth Geometric Analysis of a Tetra-Vertex Quadrilateral

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Introduction

In the realm of geometry, the study of quadrilaterals—four-sided polygons—serves as a foundational element for understanding more complex shapes and their properties. Among these, the parallelogram stands out due to its unique characteristics, such as opposite sides being parallel and equal in length. Identifying whether a given quadrilateral qualifies as a parallelogram is a classic problem that combines coordinate geometry, vector analysis, and geometric reasoning.

Today, we examine a specific quadrilateral defined by four vertices with coordinates in three-dimensional space:

- $A (1, 2, 3)$
- $B (4, 3, 1)$
- $C (2, 2, 1)$
- $D (5, 7, 3)$

Our goal is to rigorously establish whether this quadrilateral is indeed a parallelogram. This involves exploring several geometric properties and employing vector algebra to analyze the relationships between sides and diagonals.

Understanding the Geometric Context

Before delving into the calculations, it's essential to clarify the problem's scope. The vertices are given in three-dimensional space, which adds complexity compared to the two-dimensional case. The key properties we will investigate include:

- Opposite sides being parallel: For a quadrilateral to be a parallelogram, the vectors representing opposite sides must be equal or scalar multiples.
- Opposite sides being equal in length: Alternatively, the side lengths of opposite sides should be equal.
- Diagonals bisecting each other: The diagonals of a parallelogram bisect each other, meaning the midpoints of the diagonals coincide.

Our approach will systematically analyze these properties to arrive at a definitive conclusion.

Step 1: Calculating the Vectors of the Sides

The first step involves defining the vectors representing each side of the quadrilateral:

- Side AB: Vector from \(A\) to \(B\)
- Side BC: Vector from \(B\) to \(C\)
- Side CD: Vector from \(C\) to \(D\)
- Side DA: Vector from \(D\) to \(A\)

Calculations:

\[
\begin{aligned}
\vec{AB} &= \vec{B} - \vec{A} = (4 - 1, 3 - 2, 1 - 3) = (3, 1, -2) \\
\vec{BC} &= \vec{C} - \vec{B} = (2 - 4, 2 - 3, 1 - 1) = (-2, -1, 0) \\
\vec{CD} &= \vec{D} - \vec{C} = (5 - 2, 7 - 2, 3 - 1) = (3, 5, 2) \\
\vec{DA} &= \vec{A} - \vec{D} = (1 - 5, 2 - 7, 3 - 3) = (-4, -5, 0)
\end{aligned}
\]

Step 2: Testing for Parallelism of Opposite Sides

To determine if the quadrilateral is a parallelogram, the most straightforward method is to check whether the opposite sides are parallel, i.e., whether the vectors \(\vec{AB}\) and \(\vec{DC}\), and \(\vec{BC}\) and \(\vec{DA}\), are scalar multiples.

- Check \(\vec{AB}\) and \(\vec{DC}\):

\[
\vec{DC} = \vec{C} - \vec{D} = (2 - 5, 2 - 7, 1 - 3) = (-3, -5, -2)
\]

Compare with \(\vec{AB} = (3, 1, -2)\):

\[
\vec{DC} = -1 \times \vec{AB} = -1 \times (3, 1, -2) = (-3, -1, 2)
\]

But \(\vec{DC} = (-3, -5, -2)\), which is not proportional to \(\vec{AB} = (3, 1, -2)\). Therefore, sides AB and DC are not parallel.

- Check \(\vec{BC}\) and \(\vec{DA}\):

\[
\vec{DA} = (-4, -5, 0)
\]

Compare with \(\vec{BC} = (-2, -1, 0)\):

\[
\text{Is } \vec{DA} = k \times \vec{BC} \text{ for some scalar } k?
\]

Calculate k:

$$
k = \frac{-4}{-2} = 2, \quad \frac{-5}{-1} = 5, \quad \frac{0}{0} \text{ (indeterminate)}
$$

Since the ratios are inconsistent (2 vs. 5), sides BC and DA are not parallel either.

Conclusion so far: Neither pair of opposite sides is parallel, indicating that the quadrilateral is not a parallelogram based on the parallelism criterion.

Step 3: Checking the Lengths of Opposite Sides

Since parallelism doesn't hold, an alternative check is whether the lengths of opposite sides are equal, another hallmark of parallelograms.

Calculate the magnitudes:

$$
\begin{aligned}
|\vec{AB}| &= \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{9 + 1 + 4} = \sqrt{14} \\
|\vec{BC}| &= \sqrt{(-2)^2 + (-1)^2 + 0^2} = \sqrt{4 + 1 + 0} = \sqrt{5} \\
|\vec{CD}| &= \sqrt{(-3)^2 + (-5)^2 + (-2)^2} = \sqrt{9 + 25 + 4} = \sqrt{38} \\
|\vec{DA}| &= \sqrt{(-4)^2 + (-5)^2 + 0^2} = \sqrt{16 + 25 + 0} = \sqrt{41}
\end{aligned}
$$

Compare pairs:

- $|\vec{AB}| = \sqrt{14}$ versus $|\vec{CD}| = \sqrt{38}$ — not equal
- $|\vec{BC}| = \sqrt{5}$ versus $|\vec{DA}| = \sqrt{41}$ — not equal

Hence, the sides are not equal in pairs, reinforcing that the quadrilateral does not fulfill the typical properties of a parallelogram based on side lengths.

Step 4: Analyzing Diagonals for Bisection Property

Since previous tests do not suggest a parallelogram, let's examine whether the diagonals bisect each other, as this is characteristic of all parallelograms.

Identify the diagonals:

- Diagonal AC: from $A(1, 2, 3)$ to $C(2, 2, 1)$
- Diagonal BD: from $B(4, 3, 1)$ to $D(5, 7, 3)$

Calculate midpoints:

\[
\begin{aligned}
\text{Midpoint of AC} &= \left(\frac{1 + 2}{2}, \frac{2 + 2}{2}, \frac{3 + 1}{2} \right) = \left(1.5, 2, 2 \right) \\
\text{Midpoint of BD} &= \left(\frac{4 + 5}{2}, \frac{3 + 7}{2}, \frac{1 + 3}{2} \right) = \left(4.5, 5, 2 \right)
\end{aligned}
\]

Since the midpoints are different (\((1.5, 2, 2) \neq (4.5, 5, 2) \)), the diagonals do not bisect each other.

Conclusion: The diagonals do not bisect each other; hence, the quadrilateral is not a parallelogram.

Final Verdict and Insights

Having thoroughly examined the key properties—parallel sides, equal side lengths, and diagonal bisection—it is evident that the quadrilateral with vertices at:

- \(A (1, 2, 3)\)
- \(B (4, 3, 1)\)
- \(C (2, 2, 1)\)
- \(D (5, 7, 3)\)

does not exhibit the defining features of a parallelogram.

In-Depth Reflection & Contextualization

While initial intuition might suggest that such a quadrilateral could be a parallelogram, geometric analysis in three dimensions reveals otherwise. This example underscores the importance of rigorous vector algebra rather than relying solely on visual or superficial inspection—especially in three-dimensional space, where shapes can be deceptive.

Show That The Quadrilateral Having Vertices At 1 2 3 4 3 1 2 2 1 And 5 7 3 Is A Parallelogram

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