Evaluate The Line Integral C Fdr, Where F(x,y,z)=3xi+2yjzk And C Is Given By The Vector Function R(t)=sint,cost,t,

Evaluate The Line Integral C Fdr, Where F(x,y,z)=3xi+2yjzk And C Is Given By The Vector Function R(t)=sint,cost,t. This problem involves computing the line integral of a vector field along a specified curve, a fundamental concept in vector calculus with applications in physics, engineering, and mathematics. Understanding how to evaluate such integrals provides insight into the flow of vector fields, work done by forces along a path, and other physical phenomena.

In this comprehensive article, we will explore the process step-by-step, including the mathematical background, detailed calculations, and practical applications of line integrals. Whether you're a student seeking clarity or a professional refining your knowledge, this guide will serve as an invaluable resource.

Understanding the Problem: The Core Concepts

Before diving into the calculations, it’s crucial to understand the key elements involved:

What is a Line Integral?

A line integral, also known as a curve integral, evaluates the integral of a vector field F along a curve C. It essentially measures the accumulated effect of the field along that path, such as work done by a force field or the flux through a curve.

Mathematically, the line integral of a vector field F along a curve C parameterized by R(t) (with t in [a, b]) is expressed as:

$$
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{a}^{b} \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t) \, dt
$$

where:
- $\mathbf{F}(\mathbf{R}(t))$ is the vector field evaluated along the curve,
- $\mathbf{R}'(t)$ is the derivative of the parameterization, representing the tangent vector to the curve at each point.

Given Data and Problem Setup

Let's specify the elements of our problem:

- Vector Field $\mathbf{F}(x, y, z)$:

$$
\mathbf{F}(x, y, z) = 3x\, \mathbf{i} + 2y\, \mathbf{j} + z\, \mathbf{k}
$$

- Curve C Parameterized by $\mathbf{R}(t)$:

$$
\mathbf{R}(t) = \sin t\, \mathbf{i} + \cos t\, \mathbf{j} + t\, \mathbf{k}
$$

with t in an interval, typically $[0, 2\pi]$ for a full period, unless specified otherwise.

Step-by-Step Solution Approach

Evaluating the line integral involves several key steps:

1. Identify the Parameterization of the Curve

Given:
$$
\mathbf{R}(t) = \sin t\, \mathbf{i} + \cos t\, \mathbf{j} + t\, \mathbf{k}
$$
- Domain: $t \in [0, 2\pi]$, assuming a full cycle of sine and cosine.

2. Compute $\mathbf{R}'(t)$, the Derivative of $\mathbf{R}(t)$

Differentiate each component:
$$
\mathbf{R}'(t) = \frac{d}{dt} (\sin t, \cos t, t) = (\cos t, -\sin t, 1)
$$

3. Evaluate $\mathbf{F}$ Along the Curve

Substitute $x = \sin t$, $y = \cos t$, $z = t$:
$$
\mathbf{F}(\mathbf{R}(t)) = 3 (\sin t) \mathbf{i} + 2 (\cos t) \mathbf{j} + t \mathbf{k}
$$

\]

4. Compute the Dot Product \(\mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t)\)

Calculate:
\[
\left[3 \sin t,\, 2 \cos t,\, t \right] \cdot \left[\cos t,\, - \sin t,\, 1 \right]
\]
which simplifies to:
\[
(3 \sin t)(\cos t) + (2 \cos t)(- \sin t) + t \times 1
\]
\[
= 3 \sin t \cos t - 2 \sin t \cos t + t
\]
\[
= (\underbrace{3 \sin t \cos t - 2 \sin t \cos t}_{= \sin t \cos t}) + t
\]
\[
= \sin t \cos t + t
\]

5. Set Up the Integral

The line integral becomes:
\[
\int_{0}^{2\pi} \left(\sin t \cos t + t \right) dt
\]
which can be separated into two integrals:
\[
\int_{0}^{2\pi} \sin t \cos t\, dt + \int_{0}^{2\pi} t\, dt
\]

Calculating the Integrals

Integral 1: \(\int_{0}^{2\pi} \sin t \cos t\, dt\)

Recall the double-angle identity:
\[
\sin t \cos t = \frac{1}{2} \sin 2t
\]
Thus:
\[

$$
\int_{0}^{2\pi} \sin t \cos t\, dt = \frac{1}{2} \int_{0}^{2\pi} \sin 2t\, dt
$$
Evaluate:
$$
\frac{1}{2} \left[-\frac{1}{2} \cos 2t \right]_{0}^{2\pi} = -\frac{1}{4} \left[\cos 2t \right]_{0}^{2\pi}
$$
Since $\cos 2t$ is periodic with period π:
$$
\cos 2 \times 2\pi = \cos 4\pi = 1
$$
$$
\cos 0 = 1
$$
Therefore:
$$
-\frac{1}{4} (1 - 1) = 0
$$

Integral 2: $\int_{0}^{2\pi} t\, dt$

This is straightforward:
$$
\left[\frac{1}{2} t^2 \right]_0^{2\pi} = \frac{1}{2} (2\pi)^2 - 0 = \frac{1}{2} \times 4 \pi^2 = 2 \pi^2
$$

Final Result of the Line Integral

Adding the two integrals:
$$
\boxed{
\int_{C} \mathbf{F} \cdot d\mathbf{r} = 0 + 2 \pi^2 = 2 \pi^2
}
$$

This is the value of the line integral of the vector field $\mathbf{F}$ along the curve C.

Key Takeaways and Applications

Understanding how to evaluate line integrals is fundamental in multiple disciplines. Here are some key points:

- **Vector Fields and Curves:** The parameterization of the curve significantly impacts the integral's calculation and interpretation.

- **Physical Interpretation:** Line integrals often represent work done by a force field along a path or the circulation of a fluid around a loop.

- **Application in Physics:** Calculating work, analyzing electromagnetic fields, and fluid flow are common applications.

- **Mathematical Techniques:** Use identities and substitution to simplify complex integrals, as shown with the double-angle identity.

Summary

In this comprehensive guide, we evaluated the line integral of the vector field $\mathbf{F}(x, y, z) = 3x\, \mathbf{i} + 2y\, \mathbf{j} + z\, \mathbf{k}$ along the curve parameterized by $\mathbf{R}(t) = \sin t\, \mathbf{i} + \cos t\, \mathbf{j} + t\, \mathbf{k}$. The detailed steps included:

- Identifying and differentiating the parameterization.
- Substituting into the vector field.
- Computing the dot product.
- Integrating term-by-term over the interval $[0, 2\pi]$.

The result, $2 \pi^2$, provides insight into the cumulative effect of the vector field along the specified path.

Further Resources and Learning

For those interested in expanding their understanding of line integrals and vector calculus, consider exploring:

- Gradient Theorem: Simplifies certain line integrals when the field is conservative.
- Fundamental Theorem for Line Integrals: Connects potential functions

Frequently Asked Questions

What is the general approach to evaluate the line integral of a vector field F along a curve C?

The general approach involves parameterizing the curve C with a vector function $R(t)$, computing the derivative $R'(t)$, and then evaluating the integral of $F(R(t))\cdot R'(t)$ with respect to t over the given parameter interval.

Given F(x,y,z) = 3x i + 2y j + z k and R(t) = (sin t, cos t, t), how do you find F(R(t))?

Substitute $x = \sin t$, $y = \cos t$, and $z = t$ into $F(x,y,z)$: $F(R(t)) = 3(\sin t)\, i + 2(\cos t)\, j + t\, k$.

How do you compute R'(t) for the curve R(t) = (sin t, cos t, t)?

Differentiate each component with respect to t: $R'(t) = (\cos t, -\sin t, 1)$.

What is the dot product F(R(t)) · R'(t) for the given functions?

Calculate the dot product: $(3 \sin t, 2 \cos t, t) \cdot (\cos t, -\sin t, 1) = 3 \sin t \cdot \cos t + 2 \cos t \cdot (-\sin t) + t \cdot 1 = 3 \sin t \cos t - 2 \sin t \cos t + t = (3 \sin t \cos t - 2 \sin t \cos t) + t = \sin t \cos t + t$.

Over what interval should the parameter t be integrated to evaluate the line integral?

The problem specifies $R(t)$ from $t = 0$ to $t = 2\pi$, so $t \in [0, 2\pi]$.

How do you evaluate the line integral ∫_C F · dr using the parameterization?

Compute the integral $\int_0^{2\pi} (F(R(t)) \cdot R'(t))\, dt$, which in this case simplifies to $\int_0^{2\pi} [\sin t \cos t + t]\, dt$.

What is the value of the integral $\int_0^{2\pi} \sin t \cos t\, dt$?

Using the identity $\sin t \cos t = (1/2) \sin 2t$, the integral becomes $(1/2) \int_0^{2\pi} \sin 2t\, dt = (1/2) [(-1/2) \cos 2t]_0^{2\pi} = 0$.

How do you evaluate the integral $\int_0^{2\pi} t\, dt$?

The integral is $(1/2) t^2$ evaluated from 0 to 2π: $(1/2) (2\pi)^2 - 0 = (1/2) 4\pi^2 = 2\pi^2$.

What is the final value of the line integral $\int_C F \cdot dr$ for the given curve and vector field?

Adding the results: the integral of sin t cos t is 0, and the integral of t over $[0, 2\pi]$ is $2\pi^2$, so the total line integral is $0 + 2\pi^2 = 2\pi^2$.

Additional Resources

Evaluate The Line Integral C Fdr, Where F(x,y,z)=3xi+2yjzk And C Is Given By The Vector Function R(t)=sint,cost,t

Understanding the intricacies of line integrals is a fundamental aspect of vector calculus, providing crucial insights into physical phenomena such as work done by force fields, fluid flow, and electromagnetic fields. In this article, we delve into a specific problem: evaluating the line integral of the vector field F(x, y, z) = 3x i + 2y j + z k along a curve C, which is parametrized by the vector function R(t) = sin t i + cos t j + t k. This problem exemplifies the blend of geometric intuition and algebraic computation that characterizes advanced calculus, and it offers a window into the practical applications of these mathematical tools.

The Concept of Line Integrals in Vector Calculus

What Is a Line Integral?

A line integral, also known as a path integral, extends the concept of integration from functions of a single variable to vector fields along a curve in space. Intuitively, it measures the cumulative effect of a vector field F as one moves along a specified path C. For example, in physics, it can represent the work done by a force field F when moving an object along C.

Mathematically, the line integral of a vector field F along a curve C parametrized by R(t), where t varies from a to b, is expressed as:

$$
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t) dt
$$

This formulation involves substituting the parametric equations of the curve into the vector field and then computing the dot product with the derivative of the parametrization.

Applications of Line Integrals

Line integrals are instrumental across various scientific domains:

- Physics: Calculating work done by a force field, circulation in fluid flow, or electromotive force in circuits.

- Engineering: Analyzing the energy transfer along paths in systems.
- Mathematics: Studying properties of vector fields, such as conservative fields and curl.

Defining the Problem: The Vector Field and Curve

The Vector Field: $F(x, y, z) = 3x\, i + 2y\, j + z\, k$

The given vector field assigns a three-dimensional vector to each point in space:

- The x-component is $3x$, indicating the field's strength increases linearly with x.
- The y-component is $2y$, also linearly dependent on y.
- The z-component is z, proportional to the height z in space.

This field can be visualized as a spatially varying force or flow, with components that grow linearly along their respective axes.

The Curve: $R(t) = \sin t\, i + \cos t\, j + t\, k$

The curve C is parametrized by:

- $x(t) = \sin t$
- $y(t) = \cos t$
- $z(t) = t$

where t typically ranges over an interval, often from 0 to 2π for a complete revolution around a circle in the xy-plane, or over another interval depending on the problem's context.

This parametrization describes a helix winding around the z-axis, with x and y tracing a circle in the xy-plane, while z increases linearly with t. Visualizing this curve helps in understanding the path along which the vector field is integrated.

Step-by-Step Solution Approach

Step 1: Determine the Parameter Interval

To perform the integral, we need to specify the limits for t. Typically, if the problem involves a full rotation around the circle in the xy-plane, t would run from 0 to 2π.

Suppose the integral is over t from 0 to 2π, reflecting a full helix loop:

\[
t \in [0, 2\pi]
\]

Step 2: Compute the Derivative of R(t)

Differentiating R(t) component-wise:

$$
\mathbf{R}'(t) = \frac{d}{dt} (\sin t, \cos t, t) = (\cos t, -\sin t, 1)
$$

This derivative represents the tangent vector at each point on the curve, essential for the line integral.

Step 3: Evaluate the Vector Field Along the Curve

Substitute $x = \sin t$, $y = \cos t$, and $z = t$ into $F(x, y, z)$:

$$
\mathbf{F}(\mathbf{R}(t)) = 3(\sin t) i + 2(\cos t) j + t k
$$

Expressed as a vector:

$$
\mathbf{F}(\mathbf{R}(t)) = (3 \sin t, 2 \cos t, t)
$$

Step 4: Dot Product of F(R(t)) and R'(t)

Calculate:

$$
\mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t) = (3 \sin t)(\cos t) + (2 \cos t)(- \sin t) + (t)(1)
$$

Simplify each term:

- $3 \sin t \cos t$
- $- 2 \cos t \sin t$
- $+ t$

Combine the first two terms:

$$
3 \sin t \cos t - 2 \sin t \cos t = (3 - 2) \sin t \cos t = \sin t \cos t
$$

Thus, the integrand simplifies to:

$$
\sin t \cos t + t
$$

Evaluating the Integral

Step 1: Set Up the Integral

The line integral becomes:

$$
\int_{0}^{2\pi} \left(\sin t \cos t + t \right) dt
$$

This integral can be separated into two parts:

$$
\int_0^{2\pi} \sin t \cos t dt + \int_0^{2\pi} t dt
$$

Step 2: Compute the First Integral

Recall the double-angle identity:

$$
\sin t \cos t = \frac{1}{2} \sin 2t
$$

So,

$$
\int_0^{2\pi} \sin t \cos t dt = \frac{1}{2} \int_0^{2\pi} \sin 2t dt
$$

Integrate:

$$
\frac{1}{2} \left[- \frac{1}{2} \cos 2t \right]_0^{2\pi} = - \frac{1}{4} \left[\cos 2t \right]_0^{2\pi}
$$

Evaluate:

$$
- \frac{1}{4} \left(\cos 4\pi - \cos 0 \right) = - \frac{1}{4} (1 - 1) = 0
$$

since $\cos 4\pi = \cos 0 = 1$.

Step 3: Compute the Second Integral

$$
\int_0^{2\pi} t dt = \left[\frac{t^2}{2} \right]_0^{2\pi} = \frac{(2\pi)^2}{2} - 0 = \frac{4 \pi^2}{2} = 2 \pi^2
$$

Final Result:

Adding both parts:

$$
0 + 2 \pi^2 = 2 \pi^2
$$

Thus, the value of the line integral is $2\pi^2$.

Physical and Mathematical Significance

Interpreting the Result

The computed line integral, $2\pi^2$, quantifies the cumulative effect of the vector field F along the helix C. If F represented a force field, this would correspond to the work done in moving an object along the path C. The positive value indicates a net positive contribution in the direction of the path, considering the field's components.

Broader Implications

This calculation exemplifies important concepts:

- Parameterization simplifies complex spatial paths into manageable algebraic forms.
- Dot products encapsulate the interaction between the field and the path's tangent.
- Integral simplification techniques, such as using trigonometric identities, streamline the evaluation process.

Additional Considerations and Extensions

Variations in Parameter Limits

Depending on the problem context, the limits of t may vary, such as:

- From 0 to π for a semicircular path.
- Over an interval representing multiple revolutions.

Adjusting limits affects the integral's outcome, emphasizing the importance of understanding the physical or geometric scenario.

Connection to Conservative Fields

In some cases, vector fields are conservative, meaning they can be expressed as the gradient of a scalar potential. For such fields, line integrals over closed loops are zero, simplifying calculations. In this example

Evaluate The Line Integral C Fdr Where F X Y Z 3xi 2yjzk And C Is Given By The Vector Function R T Sint Cost T

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