Exercise 6.3.9: Write Down The Solution To X'' - 2x = E^(-t^2) , X(0) = 0, X'(0) = 0 As A Definite Integral.Hint:

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Understanding differential equations and their solutions is a fundamental aspect of applied mathematics and engineering. In particular, linear second-order differential equations with nonhomogeneous terms frequently appear in physics, engineering, and other sciences. Exercise 6.3.9 presents a specific second-order differential equation:

$$ X'' - 2X = e^{-t^2} $$

with initial conditions:

$$ X(0) = 0, \quad X'(0) = 0 $$

The goal is to express the solution $X(t)$ as a definite integral, thereby translating the differential problem into an integral form. This approach not only provides insight into the nature of the solution but also facilitates numerical computation and further analysis.

In this comprehensive article, we'll explore how to derive the solution to this differential equation as a definite integral, discuss the mathematical methods involved, and highlight key points for understanding and applying this approach.

Understanding the Differential Equation

Before delving into the solution process, it is essential to understand the structure of the differential equation and the significance of the initial conditions.

General Form and Types

The equation:

$$X'' - 2X = e^{-t^2}$$

is a linear, nonhomogeneous, second-order differential equation with constant coefficients. The homogeneous part:

$$X'' - 2X = 0$$

has solutions that are exponential functions, while the nonhomogeneous part e^{-t^2} is a Gaussian-like function that does not have elementary antiderivatives but can be expressed in terms of special functions like the error function.

Initial Conditions

The initial conditions:

$$X(0) = 0, \quad X'(0) = 0$$

are used to determine the constants in the general solution, ensuring the solution matches the physical or theoretical context of the problem.

Methodology for Solving the Differential Equation

The approach involves several key steps:

1. Solve the homogeneous equation.
2. Find a particular solution using an integral method.
3. Combine the solutions and apply initial conditions.
4. Express the particular solution as a definite integral for clarity and computation.

Let's explore each step in detail.

1. Homogeneous Solution

The homogeneous equation:

$$X'' - 2X = 0$$

has characteristic equation:

$$r^2 - 2 = 0 \Rightarrow r = \pm \sqrt{2}$$

Thus, the general homogeneous solution:

$$X_h(t) = C_1 e^{\sqrt{2} t} + C_2 e^{-\sqrt{2} t}$$

Applying initial conditions will later determine the constants C_1 and C_2.

2. Particular Solution via Green's Function

To find a particular solution to the nonhomogeneous equation, we employ Green's function, which transforms the differential equation into an integral form. The Green's function $G(t, s)$ for the operator $D^2 - 2$ with initial conditions at $t=0$ can be constructed from the homogeneous solutions.

The solution can be written as:

$$X(t) = \int_0^t G(t, s) e^{-s^2} ds$$

where $G(t, s)$ is the Green's function.

Constructing the Green's Function

The Green's function for a second-order linear differential operator with initial conditions at $t=0$ is typically constructed from the homogeneous solutions. For this particular operator, the Green's function can be expressed as:

$$G(t, s) = \frac{1}{W} \left[X_1(s) X_2(t) - X_2(s) X_1(t) \right]$$

where:

- $X_1(t) = e^{\sqrt{2} t}$
- $X_2(t) = e^{-\sqrt{2} t}$
- W is the Wronskian of X_1 and X_2:

$$ W = X_1(t) X_2'(t) - X_1'(t) X_2(t) $$

Since both solutions are exponential functions, the Wronskian simplifies to a constant:

$$ W = -2 \sqrt{2} $$

Using these, the Green's function becomes:

$$ G(t, s) = \frac{1}{-2 \sqrt{2}} \left[e^{\sqrt{2} s} e^{-\sqrt{2} t} - e^{-\sqrt{2} s} e^{\sqrt{2} t} \right] $$

which simplifies to:

$$ G(t, s) = \frac{1}{2 \sqrt{2}} \left[e^{-\sqrt{2}(t - s)} - e^{\sqrt{2}(t - s)} \right] $$

or equivalently,

$$ G(t, s) = - \frac{1}{\sqrt{2}} \sinh \left(\sqrt{2}(t - s) \right) $$

for $0 \leq s \leq t$.

Expressing the Solution as a Definite Integral

With the Green's function in hand, the particular solution:

$$ X_p(t) = \int_0^t G(t, s) e^{-s^2} ds $$

becomes:

$$X_p(t) = - \frac{1}{\sqrt{2}} \int_0^t \sinh \left(\sqrt{2}(t - s) \right) e^{-s^2} ds$$

This integral representation explicitly expresses $X(t)$ as a definite integral, which can be evaluated numerically or analyzed further depending on the context.

Final Solution Incorporating Initial Conditions

The general solution:

$$X(t) = X_h(t) + X_p(t)$$

becomes:

$$X(t) = C_1 e^{\sqrt{2} t} + C_2 e^{-\sqrt{2} t} - \frac{1}{\sqrt{2}} \int_0^t \sinh \left(\sqrt{2}(t - s) \right) e^{-s^2} ds$$

Applying initial conditions:

- At $t = 0$:

$$X(0) = C_1 + C_2 + 0 = 0 \Rightarrow C_1 + C_2 = 0$$

- Derivative:

$$X'(t) = C_1 \sqrt{2} e^{\sqrt{2} t} - C_2 \sqrt{2} e^{-\sqrt{2} t} - \frac{1}{\sqrt{2}} \frac{d}{dt} \int_0^t \sinh \left(\sqrt{2}(t - s) \right) e^{-s^2} ds$$

Using Leibniz's rule:

$$ \frac{d}{dt} \int_0^t \sinh \left(\sqrt{2}(t - s) \right) e^{-s^2} ds = \int_0^t \sqrt{2} \cosh \left(\sqrt{2}(t - s) \right) e^{-s^2} ds + \sinh(0) e^{-t^2} $$

which simplifies to:

$$ \sqrt{2} \int_0^t \cosh \left(\sqrt{2}(t - s) \right) e^{-s^2} ds $$

At $t=0$:

$$ X'(0) = C_1 \sqrt{2} - C_2 \sqrt{2} - 0 = 0 \Rightarrow C_1 = C_2 $$

From earlier:

$$ C_1 + C_2 = 0 \Rightarrow 2 C_1 = 0 \Rightarrow C_1 = 0 $$

and

$$ C_2 = 0 $$

Thus, the solution simplifies to:

$$ X(t) = - \frac{1}{\sqrt{2}} \int_0^t \sinh \left(\sqrt{2}(t - s) \right) e^{-s^2} ds $$

which fully satisfies the initial conditions.

Summary of the Solution as a Definite Integral

The final integral representation of the solution:

\[\boxed{

X(t) = - \frac{1}{\sqrt{2}} \int_0^t \sinh \left(\sqrt{2}(t - s) \right) e^{-s^2} ds

} \]

provides a powerful way to analyze and compute the solution to the differential equation \(X'' - 2X = e^{-t^2} \) with initial conditions \(X(0)=0 \), \(X'(0)=

Frequently Asked Questions

What is the differential equation given in Exercise 6.3.9?

The differential equation is $X'' - 2X = e^{(-t^2)}$.

What are the initial conditions specified for the problem?

The initial conditions are $X(0) = 0$ and $X'(0) = 0$.

How can the solution to the differential equation be expressed as a definite integral?

The solution can be written as an integral involving the Green's function or as an integral of the forcing function multiplied by the fundamental solutions, typically expressed as a convolution integral.

What is the significance of the hint provided in the problem?

The hint guides to express the solution as a definite integral, often involving the use of the method of variation of parameters or the Green's function approach to solve the nonhomogeneous differential equation.

Which method is most appropriate for solving this second-order linear differential equation with initial conditions?

The method of variation of parameters or the use of Green's function are appropriate for deriving the solution as a definite integral.

How is the Green's function constructed for this differential equation?

The Green's function is constructed from the fundamental solutions of the homogeneous equation X'' - 2X = 0, satisfying the initial conditions and boundary conditions as needed.

What role does the exponential function e^(-t^2) play in the integral solution?

It acts as the forcing function or source term in the differential equation, which is integrated against the Green's function to obtain the particular solution.

Can the solution be written explicitly as an integral involving special functions?

Yes, depending on the approach, the solution may involve integrals of error functions or other special functions related to the Gaussian integral, stemming from e^(-t^2).

Why is it useful to express the solution as a definite integral?

Expressing the solution as a definite integral provides an explicit formula that incorporates the initial conditions and the forcing term, facilitating analysis and numerical computation.

What is the general form of the solution to this type of differential equation as a definite integral?

The solution generally takes the form X(t) = □□^t G(t, □) e^(-□^2) d□, where G(t, □) is the Green's function for the homogeneous equation, satisfying initial conditions.

Additional Resources

Understanding the solution to Exercise 6.3.9: Write Down The Solution To X'' - 2x = E^(-t^2), X(0) = 0, X'(0) = 0 as a definite integral offers a profound insight into the power of differential equations and integral transforms. This type of problem not only highlights the methods used to solve nonhomogeneous linear differential equations but also illustrates how solutions can be expressed as integrals, providing a bridge between differential equations and integral calculus. In this comprehensive guide, we'll walk through the problem step-by-step, elucidate the underlying principles, and interpret the solution as a definite integral.

Understanding the Differential Equation and Initial Conditions

The differential equation under consideration is:

$$X'' - 2X = e^{-t^2}$$

with initial conditions:

- $X(0) = 0$
- $X'(0) = 0$

This is a second-order linear nonhomogeneous differential equation. The presence of the exponential function on the right side, e^{-t^2}, suggests that the solution involves both the homogeneous solution and a particular solution that accounts for the nonhomogeneous term.

Step 1: Solve the Homogeneous Equation

The homogeneous counterpart is:

$X'' - 2X = 0$

The characteristic equation is:

$r^2 - 2 = 0$

which yields roots:

$r = \pm\sqrt{2}$

Therefore, the homogeneous solution is:

$X_h(t) = C_1 e^{\sqrt{2} t} + C_2 e^{-\sqrt{2} t}$

where C_1 and C_2 are constants determined by initial conditions.

Step 2: Find a Particular Solution Using the Method of Variation of Parameters

Given the linearity of the differential equation, the general solution is:

$X(t) = X_h(t) + X_p(t)$

To find $X_p(t)$, particularly because the nonhomogeneous term e^{-t^2} isn't a standard function easily annihilated by characteristic solutions, variation of parameters is an effective method.

Variation of parameters involves:

- Using the homogeneous solutions $u_1(t) = e^{\sqrt{2} t}$ and $u_2(t) = e^{-\sqrt{2} t}$
- Expressing $X_p(t) = v_1(t) u_1(t) + v_2(t) u_2(t)$

The functions $v_1(t)$ and $v_2(t)$ are found by solving:

- $v_1'(t) u_1(t) + v_2'(t) u_2(t) = 0$
- $v_1'(t) u_1'(t) + v_2'(t) u_2'(t) = e^{-t^2}$

The Wronskian $W(t)$ of the fundamental solutions is:

$W(t) = u_1(t) u_2'(t) - u_1'(t) u_2(t) = -2\sqrt{2}$

which is constant with respect to t.

Solving for $v_1'(t)$ and $v_2'(t)$:

$$
v_1'(t) = -\frac{u_2(t) \cdot e^{-t^2}}{W(t)} = \frac{e^{-\sqrt{2} t} \cdot e^{-t^2}}{2 \sqrt{2}}
$$

$$
v_2'(t) = \frac{u_1(t) \cdot e^{-t^2}}{W(t)} = -\frac{e^{\sqrt{2} t} \cdot e^{-t^2}}{2 \sqrt{2}}
$$

Integrating, we get:

$$
v_1(t) = \frac{1}{2 \sqrt{2}} \int e^{-\sqrt{2} s} e^{-s^2} ds
$$

$$
v_2(t) = - \frac{1}{2 \sqrt{2}} \int e^{\sqrt{2} s} e^{-s^2} ds
$$

Thus, the particular solution is:

$$
X_p(t) = v_1(t) u_1(t) + v_2(t) u_2(t)
$$

which simplifies to:

$$
X_p(t) = \frac{e^{\sqrt{2} t}}{2 \sqrt{2}} \int_0^t e^{-\sqrt{2} s} e^{-s^2} ds - \frac{e^{-\sqrt{2} t}}{2 \sqrt{2}} \int_0^t e^{\sqrt{2} s} e^{-s^2} ds
$$

Step 3: Incorporate Initial Conditions to Find Constants

The general solution:

$$

$$
X(t) = C_1 e^{\sqrt{2} t} + C_2 e^{-\sqrt{2} t} + X_p(t)
$$

Applying initial conditions:

- At t=0:

$$
X(0) = C_1 + C_2 + X_p(0) = 0
$$

- For X'(0), differentiate:

$$
X'(t) = C_1 \sqrt{2} e^{\sqrt{2} t} - C_2 \sqrt{2} e^{-\sqrt{2} t} + X_p'(t)
$$

At t=0:

$$
X'(0) = C_1 \sqrt{2} - C_2 \sqrt{2} + X_p'(0) = 0
$$

Since $X_p(0)$ involves integrals from 0 to 0, $X_p(0) = 0$. Similarly, $X_p'(0)$ can be computed by differentiating the integral expressions and evaluating at zero, which results in zero because the integrals are zero at the lower limit.

Thus:

$$
C_1 + C_2 = 0

\]

\[

C_1 □2 - C_2 □2 = 0

\]

which implies C_1 = C_2 = 0.

Therefore, the solution simplifies to:

\[

X(t) = X_p(t)

\]

Expressing the Solution as a Definite Integral

Putting it all together, the solution X(t) is:

\[

X(t) = \frac{e^{□2 t}}{2 \sqrt{2}} \int_0^t e^{-□2 s} e^{-s^2} ds - \frac{e^{-□2 t}}{2 \sqrt{2}} \int_0^t e^{□2 s} e^{-s^2} ds

\]

This expression clearly shows the solution as a combination of definite integrals involving the exponential of quadratic functions.

Interpretation and Significance of the Solution as a Definite Integral

Expressing the solution $X(t)$ as a definite integral offers several benefits and insights:

- Explicit representation: It provides a direct formula to compute $X(t)$ without solving differential equations numerically.
- Connection to integral transforms: The integrals resemble convolution integrals found in Laplace and Fourier transforms, indicating deeper analytical structures.
- Insight into the influence of forcing functions: The integrals reflect how the nonhomogeneous term e^{-t^2} influences the solution over time.
- Foundation for numerical methods: When closed-form solutions are complicated, integral representations pave the way for numerical approximation techniques.

Additional Considerations and Techniques

While the above derivation offers a clear path, it's worth noting:

- The integrals involve the function e^{-s^2}, which is related to the error function $\text{erf}(s)$:

$$
\text{erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-u^2} du
$$

- Therefore, the integrals can be expressed in terms of erf, leading to potentially more compact formulas:

$$
\int e^{a s} e^{-s^2} ds
$$

may be expressed using the error function or its properties, especially when evaluating at specific bounds.

- Numerical evaluation of these integrals can be performed efficiently using standard libraries and algorithms that compute erf and related functions.

Summary and Final Remarks

In this detailed exploration of Exercise 6.3.9, we've demonstrated how to solve the differential equation $X'' - 2X = e^{-t^2}$ with zero initial conditions. The solution, derived via variation of parameters, elegantly expresses $X(t)$ as a combination of definite integrals involving exponential functions of quadratic forms. This integral formulation not only offers an explicit solution but also deepens our understanding of how such differential equations connect to integral calculus, special functions like the error function, and the broader landscape of applied mathematics.

This approach underscores the importance of integral representations in solving and interpreting differential equations, especially when dealing with nonhomogeneous terms that involve complex functions like e^{-t^2}. Whether for theoretical analysis or numerical computation, expressing solutions as definite integrals remains a powerful and insightful technique in the mathematician's toolkit.

Exercise 6 3 9 Write Down The Solution To X 2x E T 2 X 0 0 X 0 0 As A Definite Integral Hint

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