Find The Volume Of The Region Bounded Above By The Paraboloidz=2x2+4y2and Below By The SquareR:4x4,4y4.V=___(Simplify

Find The Volume Of The Region Bounded Above By The Paraboloid z=2x^2+4y^2 and Below By The Square R: 4x^4 + 4y^4. V=___(Simplify

Understanding the volume of complex regions bounded by different surfaces is a fundamental aspect of multivariable calculus. Such problems not only deepen our comprehension of three-dimensional space but also have practical applications in fields like physics, engineering, and computer graphics. In this article, we explore a specific problem: calculating the volume of the region bounded above by a paraboloid and below by a square-shaped boundary defined by a quartic equation. We will methodically dissect the problem, apply suitable calculus techniques, and arrive at a simplified expression for the volume.

Introduction to the Problem

Imagine a three-dimensional space where a paraboloid opens upward, and a square-shaped boundary constrains the region from below. The paraboloid is described by the surface:

$$ z = 2x^2 + 4y^2 $$

This surface curves upward, with its shape dictated by the quadratic terms in x and y. The boundary below is given by the equation:

$$ 4x^4 + 4y^4 = 1 $$

which describes a square-shaped boundary in the x-y plane, but with a quartic (fourth-degree) form rather than a simple rectangle or circle.

The goal is to find the volume V of the region enclosed between these two surfaces, i.e., the volume of the solid lying above the quartic boundary and below the paraboloid.

Understanding the Boundaries

The Paraboloid $z=2x^2+4y^2$

- Opens upward.
- The cross-section in the xy-plane is unbounded, but the boundary imposed by the quartic equation will limit the region.

The Quartic Boundary $4x^4 + 4y^4 = 1$

- Defines a closed, bounded region in the xy-plane.
- Can be rewritten as:

$$ x^4 + y^4 = \frac{1}{4} $$

- Symmetric with respect to both axes due to even powers.

Approach to the Solution

To compute the volume, we need to evaluate the double integral:

$$
V = \iint_{D} (z_{\text{top}} - z_{\text{bottom}}) \, dA
$$

where:

- $z_{\text{top}} = 2x^2 + 4y^2$
- $z_{\text{bottom}} = 0$, since the region is bounded below by the xy-plane, or the quartic boundary in the xy-plane.

However, in this problem, the lower boundary is the quartic curve $4x^4 + 4y^4 = 1$, which defines the region D in the xy-plane.

Thus, the volume can be expressed as:

$$
V = \iint_{D} (2x^2 + 4y^2) \, dA
$$

since the bottom surface is at $z=0$.

Choosing Coordinates for Simplification

The quartic boundary's complex shape suggests that standard Cartesian coordinates may complicate integration. To simplify, we look for a coordinate transformation that respects the symmetry of the boundary.

Symmetry and Substitutions

Notice the boundary equation:

$$
4x^4 + 4y^4 = 1 \quad \Rightarrow \quad x^4 + y^4 = \frac{1}{4}
$$

which is symmetric in x and y.

Because of the symmetry, it is advantageous to consider transforming the variables into new coordinates that simplify the quartic form.

Transforming the Coordinates

A promising approach involves setting:

$$
u = x^2, \quad v = y^2
$$

which are always non-negative in the region of interest. The boundary becomes:

$$
4u^2 + 4v^2 = 1 \quad \Rightarrow \quad u^2 + v^2 = \frac{1}{4}
$$

This describes a circle in the $u-v$ plane with radius $\frac{1}{2}$.

Expressing the Region in the $u-v$ Plane

- The region in the $u-v$ plane is:

$$
u \ge 0, \quad v \ge 0, \quad u^2 + v^2 \le \frac{1}{4}
$$

- The Jacobian of the transformation from (x,y) to (u,v):

$$
u = x^2 \Rightarrow du = 2x\, dx \Rightarrow dx = \frac{du}{2x}
$$

Similarly for v:

$$
v = y^2 \Rightarrow dy = \frac{dv}{2y}
$$

- The differential area element in (x,y):

$$
dA = dx\, dy
$$

Expressed in (u,v):

$$
dA = dx\, dy = \left(\frac{du}{2x}\right) \left(\frac{dv}{2y}\right) = \frac{du\, dv}{4xy}
$$

But since $x = \pm \sqrt{u}$, $y = \pm \sqrt{v}$, the Jacobian becomes more manageable if we consider only the positive quadrant (due to symmetry), where $x, y \ge 0$.

In the positive quadrant:

$$
dA = \frac{du\, dv}{4 \sqrt{u} \sqrt{v}} = \frac{du\, dv}{4 \sqrt{uv}}
$$

Formulating the Integral

The volume integral becomes:

$$
V = \iint_{D} (2x^2 + 4y^2) \, dA
$$

Expressed in terms of u and v:

$$
x^2 = u, \quad y^2 = v
$$

and

$$
2x^2 + 4y^2 = 2u + 4v
$$

The region in the $u-v$ plane is:

$$
u \ge 0, \quad v \ge 0, \quad u^2 + v^2 \le \frac{1}{4}
$$

Thus,

$$
V = \int_{v=0}^{\frac{1}{2}} \int_{u=0}^{\sqrt{\frac{1}{4} - v^2}} (2u + 4v) \times \frac{1}{4 \sqrt{uv}} \, du \, dv
$$

Simplify the integrand:

$$
V = \frac{1}{4} \int_{v=0}^{\frac{1}{2}} \int_{u=0}^{\sqrt{\frac{1}{4} - v^2}} \frac{2u + 4v}{\sqrt{uv}} \, du \, dv
$$

Note that:

$$
\frac{2u + 4v}{\sqrt{uv}} = 2u / \sqrt{uv} + 4v / \sqrt{uv} = 2 \frac{u}{\sqrt{uv}} + 4 \frac{v}{\sqrt{uv}}
$$

which simplifies to:

$$
2 \frac{\sqrt{u}}{\sqrt{v}} + 4 \frac{\sqrt{v}}{\sqrt{u}}
$$

Evaluating the Inner Integral

Expressed more clearly:

\[
V = \frac{1}{4} \int_{0}^{\frac{1}{2}} \left[\int_{0}^{\sqrt{\frac{1}{4} - v^2}} \left(2 \frac{\sqrt{u}}{\sqrt{v}} + 4 \frac{\sqrt{v}}{\sqrt{u}} \right) du \right] dv
\]

Break the integral into two parts:

\[
V = \frac{1}{4} \int_{0}^{1/2} \left[\frac{2}{\sqrt{v}}
\int_{0}^{\sqrt{\frac{1}{4} - v^2}} \sqrt{u} \, du + 4 \sqrt{v}
\int_{0}^{\sqrt{\frac{1}{4} - v^2}} \frac{1}{\sqrt{u}} \, du \right] dv
\]

Calculate each inner integral separately:

First Inner Integral

\[
\int_{0}^{a} \sqrt{u} \, du = \int_{0}^{a} u^{1/2} \, du = \frac{2}{3} u^{3/2} \bigg|_{0}^{a} = \frac{2}{3} a^{3/2}
\]

with

\[
a = \sqrt{\frac{1}{4} - v^2}
\]

Thus,

Frequently Asked Questions

How do you set up the volume integral for the region bounded above by the paraboloid $z = 2x^2 + 4y^2$ and below by the square R: $4x^4 + 4y^4$?

You set up a double integral over the square region R in the xy-plane, integrating the height difference between the paraboloid $z = 2x^2 + 4y^2$ and the xy-plane (z=0), so the volume $V = \iint_R (2x^2 + 4y^2)\, dx\, dy$.

What are the limits of integration for x and y in this volume problem where R: $4x^4 + 4y^4$?

The region R is bounded by the square where $4x^4 + 4y^4 \leq 1$, which simplifies to $x^4 + y^4 \leq \frac{1}{4}$. The limits are determined by solving these inequalities to outline the square in the xy-plane.

How can symmetry be used to simplify the calculation of this volume?

Since the region and the integrand are symmetric with respect to both axes, you can compute the volume over one quadrant and then multiply the result by 4 to find the total volume, simplifying the calculation.

What substitution or transformation can be employed to evaluate the integral over the region R?

A substitution such as $u = x^2$ and $v = y^2$ can transform the region into a more manageable shape, converting the inequality into $u^2 + v^2 \leq \frac{1}{4}$, which can simplify the integration.

How do you evaluate the double integral of $z = 2x^2 + 4y^2$ over the region R?

Express the integral as $V = \iint_R (2x^2 + 4y^2)\, dx\, dy$. Using the symmetry and substitution techniques, evaluate the integral within the bounds defined by the inequality $x^4 + y^4 \leq \frac{1}{4}$, possibly converting to polar-like coordinates or symmetry arguments.

What is the simplified expression for the volume V bounded by the paraboloid and the square R?

The exact simplified expression involves integrating over the region, resulting in $V = 4 \times \iint_{\text{quadrant}} (2x^2 + 4y^2)\, dx\, dy$, which simplifies based on the symmetry and substitution choices to an explicit numerical value.

What is the final step to compute the volume after setting up the integral?

Evaluate the double integral over the chosen region, either by direct calculation, substitution, or numerical methods, then multiply by 4 if using symmetry, to find the total volume V.

Additional Resources

Volume Calculation of the Bounded Region Between a Paraboloid and a Square

Understanding the precise volume of the space enclosed between a paraboloid and a square boundary is a classic problem in multivariable calculus, showcasing the power of coordinate transformations, symmetry, and integration techniques. In this detailed exploration, we will analyze the region defined by the paraboloid $z = 2x^2 + 4y^2$ from above, and the square $R: 4x^4 + 4y^4 \leq 1$ from below, aiming to compute its volume V.

Introduction to the Problem

Imagine a three-dimensional landscape where the surface above is a paraboloid opening upward, shaped like a bowl, described by:

$$z = 2x^2 + 4y^2.$$

Below this surface lies a square boundary, not a simple rectangle but defined implicitly by the inequality:

$$4x^4 + 4y^4 \leq 1.$$

Our goal is to find the volume of the region V bounded above by the paraboloid and below by the square R. This is a typical double integral problem, but the challenge arises from the complex boundary shape and the nonlinear nature of the boundary inequality.

Understanding the Boundaries

The Paraboloid

The surface:

$$z = 2x^2 + 4y^2$$

is a standard paraboloid opening upward. Its cross-sections:

- Along x-direction: For fixed y, the paraboloid's height varies with x as $2x^2 + 4y^2$.
- Along y-direction: For fixed x, it varies as $2x^2 + 4y^2$.

The paraboloid is symmetric about the z-axis and has a minimum at the origin \ (0, 0, 0).

The Square Boundary

The boundary is given by:

$$ 4x^4 + 4y^4 \leq 1, $$

which simplifies to:

$$ x^4 + y^4 \leq \frac{1}{4}. $$

This describes a superellipse-like boundary with a quartic form, symmetric with respect to both axes. The boundary points satisfy:

$$ x^4 + y^4 = \frac{1}{4}. $$

Because of symmetry, the region R is symmetric over both axes, which simplifies the integration process.

Transforming the Boundary for Simplification

Symmetry Considerations

Given the symmetry of the boundary $x^4 + y^4 \leq 1/4$, the region can be confined to the first quadrant and then multiplied by 4 to account for symmetry:

$$
V = 4 \times \text{volume over } x \ge 0, y \ge 0.
$$

Boundary in the First Quadrant

In the first quadrant:

$$
x, y \ge 0, \quad x^4 + y^4 \leq \frac{1}{4}.
$$

The boundary curve is:

$$
x^4 + y^4 = \frac{1}{4}.

\]

Change of Variables to Simplify the Boundary

To handle the quartic boundary, a substitution that simplifies \(x^4 + y^4 \) is beneficial. Consider the following:

\[
u = x^2, \quad v = y^2,
\]

which converts the boundary:

\[
u^2 + v^2 \leq \frac{1}{4}.
\]

This is because:

\[
x^4 = (x^2)^2 = u^2,
\]
\[
y^4 = v^2,
\]
and the boundary becomes:

\[
u^2 + v^2 \leq \frac{1}{4}.
\]

Note: Since \(x, y \ge 0 \), then \(u, v \ge 0 \). The boundary in the \((u, v) \)-plane is a quarter circle of radius \(1/2 \):

\[
u^2 + v^2 \leq \left(\frac{1}{2}\right)^2.
\]

Expressing the Volume Integral

Setting up the Integral in \((x, y) \)-coordinates

The volume:

\[

V = \iint_{R} z \, dA,
\]

where \(R \) is the region bounded by \(4x^4 + 4y^4 \leq 1 \), and \(z = 2x^2 + 4y^2 \).

Using symmetry, the volume can be written as:

\[
V = 4 \times \iint_{x \ge 0, y \ge 0, x^4 + y^4 \leq \frac{1}{4}} (2x^2 + 4y^2) \, dx \, dy.
\]

Switching to the \((u, v) \) variables:

\[
x = \sqrt{u}, \quad y = \sqrt{v},
\]

then:

\[
dx = \frac{1}{2 \sqrt{u}} du, \quad dy = \frac{1}{2 \sqrt{v}} dv,
\]

and the Jacobian of the transformation:

\[
dx\,dy = \frac{1}{4 \sqrt{u v}} du\, dv.
\]

Expressed in \((u, v) \):

\[
z = 2x^2 + 4y^2 = 2u + 4v,
\]

and the integral becomes:

\[
V = 4 \times \iint_{u,v \ge 0, u^2 + v^2 \leq \frac{1}{4}} (2u + 4v) \times \frac{1}{4 \sqrt{u v}} du\, dv.
\]

Simplify:

\[
V = \iint_{u,v \ge 0, u^2 + v^2 \leq \frac{1}{4}} \frac{(2u + 4v)}{\sqrt{u v}} du\, dv.
\]

Switching to Polar Coordinates in the (u, v)-plane

Since the boundary in (u, v)-space is a quarter circle of radius $1/2$:

$$
u = r \cos \theta, \quad v = r \sin \theta, \quad r \in [0, 1/2], \quad \theta \in [0, \pi/2].
$$

The Jacobian for this transformation:

$$
du\, dv = r dr d\theta.
$$

Express the integrand:

$$
\frac{2u + 4v}{\sqrt{u v}} = \frac{2r \cos \theta + 4r \sin \theta}{\sqrt{r \cos \theta \times r \sin \theta}} = \frac{r (2 \cos \theta + 4 \sin \theta)}{r \sqrt{\cos \theta \sin \theta}} = \frac{2 \cos \theta + 4 \sin \theta}{\sqrt{\cos \theta \sin \theta}}.
$$

Thus, the volume integral becomes:

$$
V = \int_{0}^{\pi/2} \int_{0}^{1/2} \frac{2 \cos \theta + 4 \sin \theta}{\sqrt{\cos \theta \sin \theta}} \times r dr d\theta.
$$

Evaluating the Integral

Integration over r

The r-integral:

$$
\int_{0}^{1/2} r dr = \frac{(1/2)^2}{2} = \frac{1/4}{2} = \frac{1}{8}.
$$

```
Remaining integral over \( \theta \)

The volume simplifies to:

\[
V = \frac{1}{8} \int_{0}^{\pi/2} \frac{2 \cos \theta + 4 \sin
\theta}{\sqrt{\cos \theta \sin \theta}} d\theta.
\]

Express numerator:

\[
2 \cos \theta + 4 \sin \theta.
\]

Express denominator:

\[
\sqrt{\cos \theta \sin \theta} = (\cos \theta \sin \theta)^{1/2}.
\]

Rewrite the integral:

\[
V = \frac{1}{8} \int_{0}^{\pi/2} \left( 2 \frac{\cos \theta}{(\cos \theta
\sin \theta)^{1/2}} + 4 \frac{\sin \theta}{(\cos \theta \sin \theta)^{1/2}}
\right) d\theta.
\]

Split into two integrals:

\[
V = \frac{1}{8} \left[ 2 \int_{0}^{\pi/2} \frac{\cos \theta}{(\cos
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