q2Find The Volume Of The Region Bounded Above By The Elliptical Paraboloid Z-8-3x - 4y And Below By"

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Understanding how to calculate the volume of regions bounded by surfaces is a fundamental aspect of multivariable calculus. In particular, regions bounded above by an elliptical paraboloid and below by a different surface or plane are common in various applications, including physics, engineering, and mathematical modeling. In this article, we will explore a detailed, step-by-step approach to finding the volume of the region bounded above by the elliptical paraboloid $z = 8 - 3x - 4y$ and below by another surface or plane—though the precise lower boundary was not fully provided. For the purpose of this discussion, we will assume the lower boundary is the plane $z = 0$, which is a common scenario in volume calculations.

Our goal is to understand how to set up and evaluate the integral that represents this volume, using techniques such as coordinate transformations, especially converting to polar or elliptical coordinates, to simplify the process.

Understanding the Surfaces and Boundaries

The Elliptical Paraboloid $z = 8 - 3x - 4y$

The surface described by the equation $z = 8 - 3x - 4y$ is an elliptical paraboloid opening downward, shifted such that the vertex is at the point where z attains its maximum. The key features include:

- Intercepts: When $x = 0$ and $y = 0$, $z = 8$.
- Intercepts with axes:
- For $y = 0$: $z = 8 - 3x$
- For $x = 0$: $z = 8 - 4y$

- Axis behavior: The surface slopes downward as x and y increase.

The surface's level curves (curves of constant z) are ellipses, which makes it elliptical paraboloid.

The Lower Boundary: The Plane $z = 0$

Most volume calculations involve the region between a surface and a plane. Assuming the lower boundary is the plane $z = 0$, the problem reduces to finding the volume between the paraboloid and the xy-plane over the domain where the paraboloid is above $z=0$.

Setting Up the Volume Integral

The volume V of the region bounded above by the paraboloid $z = 8 - 3x - 4y$ and below by $z=0$ is given by the double integral:

$$
V = \iint_{D} [(8 - 3x - 4y) - 0] \, dA
$$

where D is the projection of the region onto the xy-plane, i.e., the set of points (x,y) satisfying:

$$
8 - 3x - 4y \geq 0
$$

which simplifies to:

$$
3x + 4y \leq 8
$$

and $x, y \geq 0$ if we are only considering the first quadrant.

Finding the Projection Domain D

The inequality $3x + 4y \leq 8$ defines a bounded region in the xy-plane, specifically a triangle with vertices at:

- When $y=0$: $x = \frac{8}{3}$
- When $x=0$: $y = 2$
- The intercepts: $(\frac{8}{3}, 0)$ and $(0, 2)$

The region D can be described as:

$$
D = \left\{ (x,y) \mid x \geq 0, y \geq 0, 3x + 4y \leq 8 \right\}
$$

which is a right triangle in the first quadrant.

Choosing an Appropriate Coordinate System

The linear boundary $3x + 4y = 8$ suggests that a change of variables might simplify the integral. A common approach is to convert to variables aligned with the boundary, such as:

- Standard Cartesian Coordinates: straightforward but may require more work.
- Change of Variables (Transformation): to simplify the boundary into a rectangle.

Linear Transformation to Simplify the Domain

Define new variables:

$$
u = 3x + 4y \\
v = \text{(another linear combination orthogonal to } u)
$$

But since the boundary is linear, a more straightforward approach is to set the limits directly in Cartesian coordinates:

- For each fixed y in $[0, 2]$:
- x varies from 0 to $\frac{8 - 4y}{3}$.

Thus, the integral becomes:

$$
V = \int_{y=0}^{2} \int_{x=0}^{\frac{8 - 4y}{3}} (8 - 3x - 4y) \, dx \, dy
$$

Evaluating the Double Integral

Step-by-step, the process involves:

1. Inner integral: with respect to \(x \):

\[
I(y) = \int_{0}^{\frac{8 - 4y}{3}} (8 - 3x - 4y) \, dx
\]

2. Outer integral: with respect to \(y \):

\[
V = \int_{0}^{2} I(y) \, dy
\]

Calculating the Inner Integral

Compute:

\[
I(y) = \int_{0}^{\frac{8 - 4y}{3}} (8 - 3x - 4y) \, dx
\]

Treating \(y \) as a constant:

\[
I(y) = \left[8x - \frac{3}{2}x^2 - 4yx \right]_0^{\frac{8 - 4y}{3}}
\]

Evaluate at the upper limit:

\[
x = \frac{8 - 4y}{3}
\]

\[
I(y) = 8 \left(\frac{8 - 4y}{3} \right) - \frac{3}{2} \left(\frac{8 - 4y}{3} \right)^2 - 4y \left(\frac{8 - 4y}{3} \right)
\]

Simplify each term:

- First term:

\[

\frac{8(8 - 4y)}{3} = \frac{64 - 32y}{3}
\]

- Second term:

\[
\frac{3}{2} \times \frac{(8 - 4y)^2}{9} = \frac{3}{2} \times \frac{(8 -
4y)^2}{9} = \frac{(8 - 4y)^2}{6}
\]

- Third term:

\[
\frac{4y(8 - 4y)}{3}
\]

Putting it all together:

\[
I(y) = \frac{64 - 32y}{3} - \frac{(8 - 4y)^2}{6} - \frac{4y(8 - 4y)}{3}
\]

Simplifying the Expression

Compute \((8 - 4y)^2 \):

\[
(8 - 4y)^2 = 64 - 64y + 16 y^2
\]

Now, rewrite \(I(y) \):

\[
I(y) = \frac{64 - 32 y}{3} - \frac{64 - 64 y + 16 y^2}{6} - \frac{4 y (8 - 4
y)}{3}
\]

Next, simplify the third term:

\[
4 y (8 - 4 y) = 32 y - 16 y^2
\]

So,

\[
I(y) = \frac{64 - 32 y}{3} - \frac{64 - 64 y + 16 y^2}{6} - \frac{32 y - 16

y^2}{3}
\]

Express all over common denominators:

\[
I(y) = \frac{2(64 - 32 y)}{6} - \frac{64 - 64 y + 16 y^2}{6} - \frac{2(32 y - 16 y^2)}{6}
\]

Compute each numerator:

- First term numerator:

\[
2(64 - 32 y) = 128 - 64 y
\]

- Third term numerator:

\[
2(32 y - 16 y^2) = 64 y - 32 y^2
\]

Now, rewrite:

\[
I(y) = \frac{128 - 64 y - (64 - 64 y +

Frequently Asked Questions

How do I set up the triple integral to find the volume of the region bounded above by the elliptical paraboloid $z = 8 - 3x - 4y$?

To find the volume, express the region in the xy-plane where the paraboloid exists, then set up a double integral over that region with the integrand being the height difference (from the xy-plane to the paraboloid). Specifically, since z varies from the xy-plane ($z=0$) up to $z=8 - 3x - 4y$, the volume $V = \iint_D (8 - 3x - 4y)\, dx\, dy$, where D is the projection of the paraboloid onto the xy-plane.

What is the projection region D in the xy-plane for the given paraboloid?

The projection region D is determined by the set of (x,y) satisfying $z \geq 0$, so $8 - 3x - 4y \geq 0$, which simplifies to $3x + 4y \leq 8$. The region D is thus the set of points (x,y) in the plane satisfying $3x + 4y \leq 8$ with $x, y \geq 0$ if the paraboloid is bounded in the positive quadrant; otherwise, it includes all (x,y) satisfying the inequality, potentially extending into all quadrants depending on the context.

How can I convert the double integral to polar coordinates to simplify the calculation?

Since the boundary is linear ($3x + 4y \leq 8$), converting to polar coordinates ($x = r \cos \theta$, $y = r \sin \theta$) can simplify the region. The inequality becomes $3r \cos \theta + 4r \sin \theta \leq 8$, or $r (3 \cos \theta + 4 \sin \theta) \leq 8$. For each θ, r varies from 0 up to $r = 8 / (3 \cos \theta + 4 \sin \theta)$, provided the denominator is positive. You then set up the integral over θ from 0 to 2π and r from 0 to that upper limit.

What are the steps to evaluate the volume integral once the region is set up?

First, express the region in suitable coordinates (Cartesian or polar). Then, write the double integral of the paraboloid's height function over the projected region. Next, perform the integration with respect to y (or r in polar) first, followed by x (or θ), simplifying step-by-step. Use substitution if necessary to evaluate the resulting integrals, and compute the definite integrals to find the volume.

Are there any special considerations or common mistakes when calculating the volume bounded by an elliptical paraboloid?

Yes, common mistakes include misidentifying the projection region D, forgetting to set the correct limits of integration, or mishandling the inequalities that define the projection. Also, ensure the integrand correctly represents the height difference, and be cautious when converting to polar coordinates, especially regarding the bounds and the sign of the denominator. Double-check the region's shape and the bounds before integrating.

Can symmetry be used to simplify the calculation of the volume in this problem?

Yes, if the paraboloid and the projection region are symmetric about certain axes, symmetry can significantly simplify the integral. For example, if the region is symmetric about the x- or y-axis, you may integrate over a quarter segment and multiply the result accordingly. However, in this case, since the boundary is linear and not symmetric about the axes, symmetry may be limited, so carefully analyze the region to identify any exploitable symmetry.

What is the final formula for the volume of the region bounded above by the paraboloid z = 8 - 3x - 4y and below by the xy-plane?

The volume V is given by the double integral over the projection region D:

$V = \iint_D (8 - 3x - 4y)\, dx\, dy$,

where D is defined by $3x + 4y \leq 8$ with $x, y \geq 0$ (or the appropriate bounds depending on the context). Alternatively, converting to polar coordinates, the integral becomes:

$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{8/(3 \cos \theta + 4 \sin \theta)} (8 - 3r \cos \theta - 4r \sin \theta)\, r\, dr\, d\theta$,

which can be evaluated step-by-step to find the volume.

Additional Resources

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Introduction

The calculation of the volume of a region bounded by a surface and a plane or other surfaces is a fundamental task in multivariable calculus, with applications spanning physics, engineering, and mathematical modeling. Among the myriad of surfaces studied, paraboloids—particularly elliptical paraboloids—are prominent due to their symmetry and occurrence in natural phenomena and design.

In this investigation, we focus on a specific problem: "q2Find The Volume Of The Region Bounded Above By The Elliptical Paraboloid Z = 8 - 3x - 4y". This problem encapsulates the core challenges of setting up and evaluating triple integrals, understanding geometric boundaries, and applying coordinate transformations.

This article aims to thoroughly dissect this problem, present the methods involved in solving it, and explore the broader implications and techniques pertinent to such volume calculations.

Understanding the Geometric Setting

The Elliptical Paraboloid

The surface in question is described by the equation:

$$ Z = 8 - 3x - 4y $$

This is a plane, but the problem states it as an elliptical paraboloid, which suggests a potential misstatement or a need for clarification. Typically, an elliptical paraboloid has the form:

$$ Z = \frac{x^2}{a^2} + \frac{y^2}{b^2} $$

or

$$ Z = c - \alpha x^2 - \beta y^2 $$

Given the provided linear form $Z = 8 - 3x - 4y$, it appears to be a plane, not a paraboloid.

However, if the original statement intends to refer to a paraboloid, perhaps the equation is:

$$ Z = 8 - 3x^2 - 4y^2 $$

which is an elliptical paraboloid opening downward with its vertex at $(0, 0, 8)$.

Assumption Clarification:

Based on the typical structure of such problems, and the phrase "bounded above by the elliptical paraboloid," it is likely that the intended surface is:

$$ Z = 8 - 3x^2 - 4y^2 $$

which is a downward-opening elliptical paraboloid.

The Region Bounded Below

The problem states that the region is bounded below by some other surface or plane, but the statement is incomplete. Usually, such problems involve the paraboloid and a plane or the xy-plane (i.e., $Z = 0$).

Assumption:

The region is bounded above by the elliptical paraboloid:

$$ Z = 8 - 3x^2 - 4y^2 $$

and below by the plane:

$$ Z = 0 $$

which is the xy-plane.

Thus, the volume V we seek is the volume between the paraboloid and the xy-plane, over the region where the paraboloid is above the xy-plane.

Setting Up the Problem

Step 1: Find the projection of the volume onto the xy-plane

The volume is the set of points (x, y, z) satisfying:

$$ 0 \leq z \leq 8 - 3x^2 - 4y^2 $$

with the constraints:

$$ z \geq 0 $$
$$ 8 - 3x^2 - 4y^2 \geq 0 $$

which simplifies to:

$$ 3x^2 + 4y^2 \leq 8 $$

This inequality defines an elliptical region in the xy-plane.

Step 2: Describe the region in xy-plane

The boundary of the projection region is:

$$ 3x^2 + 4y^2 = 8 $$

which is an ellipse centered at the origin.

Expressed explicitly:

$$ \frac{x^2}{8/3} + \frac{y^2}{2} = 1 $$

Thus, the projection region R is an ellipse with semi-axes:

$$ a = \sqrt{\frac{8}{3}} $$
$$ b = \sqrt{2} $$

Calculating the Volume

Step 3: Set up the double integral

The volume is:

$$
V = \iint_{R} (8 - 3x^2 - 4y^2) \, dA
$$

where R is the elliptical region defined above.

Step 4: Coordinate transformation

To evaluate the integral efficiently, it is common to apply a change of variables to convert the elliptical region into a circle.

Define:

$$
u = \frac{x}{a} = \frac{x}{\sqrt{8/3}} \\
v = \frac{y}{b} = \frac{y}{\sqrt{2}}
$$

then:

$$
x = a u = \sqrt{\frac{8}{3}} \, u \\
y = b v = \sqrt{2} \, v
$$

The Jacobian determinant of this transformation is:

$$
J = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = a \times b =
\sqrt{\frac{8}{3}} \times \sqrt{2} = \sqrt{\frac{8}{3} \times 2} =
\sqrt{\frac{16}{3}} = \frac{4}{\sqrt{3}}
$$

The region R in uv-space is the unit disk:

$$
u^2 + v^2 \leq 1
$$

Step 5: Rewrite the integrand

Express $8 - 3x^2 - 4y^2$ in terms of u and v:

$$
8 - 3x^2 - 4y^2 = 8 - 3(a u)^2 - 4 (b v)^2
$$

Calculate:

$$
3a^2 u^2 = 3 \times \frac{8}{3} u^2 = 8 u^2

\]
\[
4b^2 v^2 = 4 \times 2 v^2 = 8 v^2
\]

Therefore,

\[
8 - 3x^2 - 4y^2 = 8 - 8 u^2 - 8 v^2 = 8(1 - u^2 - v^2)
\]

Final Integral Expression

Putting everything together, the volume integral becomes:

\[
V = \iint_{u^2 + v^2 \leq 1} 8(1 - u^2 - v^2) \times |J| \, du \, dv
\]

where \(|J| = \frac{4}{\sqrt{3}} \).

Expressed explicitly:

\[
V = \frac{4}{\sqrt{3}} \times 8 \iint_{u^2 + v^2 \leq 1} (1 - u^2 - v^2) \, du \, dv
\]

which simplifies to:

\[
V = \frac{32}{\sqrt{3}} \iint_{u^2 + v^2 \leq 1} (1 - u^2 - v^2) \, du \, dv
\]

Evaluation Using Polar Coordinates

Step 6: Switch to polar coordinates

In the \(uv \)-plane, the region is the unit disk:

\[
u = r \cos \theta, \quad v = r \sin \theta, \quad r \in [0,1], \quad \theta \in [0, 2\pi]
\]

The Jacobian for polar coordinates is \(r \).

The integrand becomes:

$$
1 - r^2
$$

Thus,

$$
V = \frac{32}{\sqrt{3}} \int_{0}^{2\pi} \int_{0}^{1} (1 - r^2) r \, dr \, d\theta
$$

Computing the Integral

Step 7: Integrate with respect to r

$$
\int_{0}^{1} (1 - r^2) r \, dr = \int_{0}^{1} (r - r^3) \, dr
$$

Calculate:

$$
\int_{0}^{1} r \, dr = \frac{1}{2}
$$

$$
\int_{0}^{1} r^3 \, dr = \frac{1}{4}
$$

So,

$$
\int_{0}^{1} (r - r^3) \, dr = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}
$$

Step 8: Integrate with respect to θ

$$
\int_{0}^{2\pi} d\theta = 2\pi
$$

Step 9: Final expression for volume

Putting it all together:

$$
V = \frac{32}{\sqrt{3}} \times 2\pi
$$

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