Suppose X And Y Have Joint Density F(x,y).Are X And Y Independent Ifa. F(x,y)=xe^x(1+y) For X,y0?b. F(x,y)=6xy^2

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Introduction

Understanding the concepts of joint density functions and independence between random variables is fundamental in probability theory and statistics. When analyzing two continuous random variables, X and Y, their joint density function, denoted as F(x,y), describes the likelihood of observing specific pairs (x, y). Determining whether X and Y are independent involves examining whether the joint density can be expressed as a product of their marginal densities.

This article explores the criteria for independence in the context of given joint density functions. Specifically, we analyze two scenarios:

1. When $F(x,y) = xe^x(1 + y)$ for $x, y \geq 0$.
2. When $F(x,y) = 6xy^2$ for $x, y \geq 0$.

By systematically calculating marginal densities, verifying the properties of joint densities, and examining whether the joint density factors into the product of marginals, we aim to answer the question: Are X and Y independent in these cases?

Understanding Joint Density and Independence

Joint Density Function

A joint density function, f(x, y), of two continuous random variables X and Y, must satisfy two main properties:

- Non-negativity: $f(x, y) \geq 0$ for all (x, y).
- Total probability integral: $\iint f(x, y)\, dx\, dy = 1$ over the support.

The joint density encapsulates the probability distribution of X and Y together, providing the likelihood of their simultaneous outcomes.

Independence of Random Variables

Two continuous random variables X and Y are independent if and only if their joint density factors into the product of their marginal densities:

$$ f(x, y) = f_X(x) \times f_Y(y) $$

where:

- $f_X(x) = \int f(x, y) dy$
- $f_Y(y) = \int f(x, y) dx$

If this factorization holds, knowing the value of X provides no information about Y, and vice versa.

Case 1: Analyzing F(x,y) = xe^x(1 + y) for x, y ≥ 0

Step 1: Verify if F(x,y) is a valid joint density

- Non-negativity: Since x ≥ 0, e^x > 0, and (1 + y) ≥ 1 for y ≥ 0, the function $f(x, y) = x e^x (1 + y)$ is non-negative over the support.

- Total probability: Check if the integral over the support equals 1:

$$
\int_{x=0}^\infty \int_{y=0}^\infty x e^x (1 + y) dy dx
$$

Compute the inner integral:

$$
\int_{y=0}^\infty (1 + y) dy = \left[y + \frac{y^2}{2} \right]_0^\infty
$$

Since the upper limit tends to infinity, the integral diverges (goes to infinity). Therefore, $f(x, y)$ does not integrate to 1 over the support, implying it is not a valid joint density.

Conclusion: Because the joint density is not normalized (integral diverges), $F(x,y) = x e^x(1 + y)$ is not a valid joint density function, and the question of independence is moot in this case.

Note: For a properly defined density, the integral must converge to 1. Since it doesn't here, this scenario is invalid, but for the sake of understanding, we proceed to examine the structure.

Step 2: Attempt to find marginal densities (hypothetically)

Suppose, hypothetically, the normalization constant is adjusted to make $f(x, y)$ a proper density. Then,

- Marginal of X:

$$
f_X(x) = \int_0^\infty f(x, y) dy = x e^x \int_0^\infty (1 + y) dy
$$

- Marginal of Y:

$$
f_Y(y) = \int_0^\infty f(x, y) dx = (1 + y) \int_0^\infty x e^x dx
$$

However, both integrals diverge (since integrating y or $x e^x$ over infinite support does not converge). Therefore, the marginal densities are not finite, further confirming the invalidity of this joint density.

Final Assessment:
Because the joint density is invalid (not integrable), the question of independence does not arise. The function does not define a valid probability distribution over the support.

Case 2: Analyzing $F(x,y) = 6xy^2$ for $x, y \geq 0$

Step 1: Verify if $F(x,y)$ is a valid joint density

- Non-negativity: Since $x \geq 0$ and $y \geq 0$, $6xy^2 \geq 0$.

- Total probability:

Calculate:

$$
\int_{x=0}^\infty \int_{y=0}^\infty 6xy^2 dy dx
$$

Inner integral over y:

$$
\int_0^\infty y^2 dy
$$

which diverges to infinity. To make the integral finite, the support must be bounded or the density function must include a normalization constant that ensures the total integral equals 1.

Suppose the support is finite, e.g., $0 \leq x \leq 1$ and $0 \leq y \leq 1$:

$$
\int_{x=0}^1 \int_{y=0}^1 6xy^2 dy dx
$$

Calculate the inner integral:

$$
\int_0^1 y^2 dy = \frac{1}{3}
$$

Then,

$$
\int_0^1 6x \times \frac{1}{3} dx = 2 \int_0^1 x dx = 2 \times \frac{1}{2} = 1
$$

Thus, with support $[0,1] \times [0,1]$, $f(x,y) = 6xy^2$ is a valid joint density.

Assumption: For the analysis, we adopt this bounded support, making the density valid.

Step 2: Find the marginal densities

- Marginal of X:

$$
f_X(x) = \int_0^1 6 xy^2 dy = 6x \int_0^1 y^2 dy = 6x \times \frac{1}{3} = 2x
$$

for $0 \leq x \leq 1$.

- Marginal of Y:

$$
f_Y(y) = \int_0^1 6 xy^2 dx = 6 y^2 \int_0^1 x dx = 6 y^2 \times \frac{1}{2} = 3 y^2
$$

for $0 \leq y \leq 1$.

Step 3: Check if the joint density factors into marginals

Recall the joint density:

$$
f(x, y) = 6 xy^2
$$

and the marginals:

$$
f_X(x) = 2x \quad \text{and} \quad f_Y(y) = 3 y^2
$$

Compute the product:

$$
f_X(x) \times f_Y(y) = (2x) \times (3 y^2) = 6 xy^2
$$

which matches the joint density function.

Conclusion:
Since the joint density equals the product of the marginals, X and Y are independent in this case.

Summary and Final Insights

- In the first scenario, $F(x,y) = xe^x(1 + y)$ is not a valid joint density function because its integral over the support diverges. Therefore, questions of independence do not apply.

- In the second scenario, assuming a bounded support such as $[0,1] \times [0,1]$, the joint density function $F(x,y) = 6 xy^2$ is valid. Its marginals are $f_X(x) = 2x$ and $f_Y(y) = 3 y^2$, and their product reproduces the joint density exactly. Hence, X and Y are independent in this case.

Practical Implications in Probability and Statistics

Understanding the independence of random variables is essential in various applications:

- Model Simplification: Independence allows for the joint distribution to be expressed as a product of marginals, simplifying calculations.
- Data Analysis: Recognizing independence helps in modeling and inference, such as in regression, hypothesis testing, and Bayesian analysis

Frequently Asked Questions

How can we determine if two random variables X and Y are independent given their joint density function?

X and Y are independent if and only if their joint density function factors into the product of their marginal densities, i.e., $F(x,y) = f_X(x) f_Y(y)$ for all x, y.

Given the joint density $F(x,y) = x e^{x} (1 + y)$ for $x, y \geq 0$, are X and Y independent?

To check independence, verify if $F(x,y) = f_X(x) f_Y(y)$. Since $F(x,y)$ does not factor into functions solely of x and y separately, X and Y are not independent.

For the joint density $F(x,y) = 6xy^2$ defined for $x, y \geq 0$, are X and Y independent?

Yes. Since $F(x,y) = 6xy^2$ can be expressed as the product of $f_X(x) = 6x$ and $f_Y(y) = y^2$, X and Y are independent.

How do you compute the marginal densities from a joint density function to check for independence?

The marginal density of X is found by integrating the joint density over all y, and similarly for Y by integrating over x. If the joint density equals the product of these marginals everywhere, the variables are independent.

What are common mistakes to avoid when testing independence from joint density functions?

A common mistake is assuming independence without verifying if the joint density factors appropriately. Always compute marginals and check if the joint equals their product across the domain.

Can a joint density function be valid if it doesn't factor into the product of marginal densities? What does this imply about independence?

Yes, a joint density can be valid without factoring into the product of marginals. This indicates that the variables are dependent.

Additional Resources

Understanding the Independence of Random Variables X and Y Through Joint Density Functions

In probability theory and statistics, the concept of independence between random variables plays a fundamental role in analyzing complex systems and phenomena. When two random variables, X and Y, are said to be independent, their joint behavior can be completely characterized by their individual behaviors, simplifying many probabilistic computations and interpretations. A key question often posed is: Given the joint density function \(f_{X,Y}(x,y) \), are X and Y independent? This article delves into this question through the lens of two specific joint density functions, exploring the criteria for independence, the calculations involved, and the implications of these findings.

Fundamentals of Independence in Continuous Random Variables

Definition of Independence

Two continuous random variables X and Y are independent if and only if their joint probability density function (pdf) factorizes into the product of their marginal densities:

\[
f_{X,Y}(x,y) = f_X(x) \times f_Y(y)

$$

for all points (x,y) within their joint support. This condition implies that knowing the value of one variable provides no information about the other.

Implications of Independence

- Simplified calculations: The joint distribution simplifies to the product of marginals, easing the computation of probabilities.
- Predictive modeling: Independence assumptions are central in many models, such as Bayesian networks, simplifying complex dependencies.
- Statistical tests: Determining independence is crucial in hypothesis testing, including tests like Chi-square or correlation tests.

Methodology to Test Independence

Given a joint density $f_{X,Y}(x,y)$:

1. Compute the marginal densities:
$$
f_X(x) = \int_{-\infty}^\infty f_{X,Y}(x,y) dy
$$
$$
f_Y(y) = \int_{-\infty}^\infty f_{X,Y}(x,y) dx
$$
2. Verify the factorization:
Check whether $f_{X,Y}(x,y) = f_X(x) \times f_Y(y)$ holds for all (x,y) in the joint support.

Case 1: Joint Density $f(x,y) = x e^x (1 + y)$ for $x, y \geq 0$

Understanding the Density Function

The provided joint density function is:

$$
f_{X,Y}(x,y) = x e^x (1 + y), \quad x \geq 0, y \geq 0
$$

This density suggests a dependence structure that explicitly involves the product of functions of x and y, which raises questions about independence.

Step 1: Verify if $f_{X,Y}(x,y)$ is a valid density

To confirm the validity, the total probability must equal 1:

$$
\int_0^\infty \int_0^\infty f_{X,Y}(x,y) dy dx = 1
$$

Calculating the integral:

$$
\int_0^\infty \left(x e^x \int_0^\infty (1 + y) dy \right) dx
$$

However, note that $\int_0^\infty (1 + y) dy$ diverges to infinity. This indicates that the density as specified is not properly normalized or may be incomplete.

Correction: To ensure the density is valid, it must be integrable over its support. Possibly, the original density is intended with some normalization constants or limited support in y.

Assuming the density is truncated or normalized correctly, or perhaps the density function is:

$$
f_{X,Y}(x,y) = c \times x e^x (1 + y), \quad x \geq 0, y \in [0, Y_{max}]
$$
with a normalization constant c.

For demonstration purposes, assume that the support for y is limited to a finite interval, say $0 \leq y \leq Y_{max}$, and the density is normalized accordingly.

Key Point: The main focus is on whether the joint density factorizes, not on the normalization constants.

Step 2: Find the marginal densities

- Marginal of X:

$$

$$
f_X(x) = \int_0^{Y_{max}} f_{X,Y}(x,y) dy = x e^x \int_0^{Y_{max}} (1 + y) dy
$$

$$
= x e^x \left[y + \frac{y^2}{2} \right]_0^{Y_{max}} = x e^x \left(Y_{max} + \frac{Y_{max}^2}{2} \right)
$$

- Marginal of Y:

$$
f_Y(y) = \int_0^{\infty} f_{X,Y}(x,y) dx = (1 + y) \int_0^{\infty} x e^x dx
$$

But $\int_0^{\infty} x e^x dx$ diverges (since e^x grows exponentially). This suggests the original density is not integrable over the entire support unless there's an exponential decay factor or bounded support.

Conclusion:

Without proper normalization or bounded support, the density cannot be valid. However, assuming the density is normalized over a finite support, or with proper decay, the key point is to analyze whether the joint density factorizes into functions of x and y.

Step 3: Check for factorization (Independence)

Suppose the marginal densities are:

$$
f_X(x) = A x e^x
$$

$$
f_Y(y) = B (1 + y)
$$

for some constants A and B, with the joint density:

$$
f_{X,Y}(x,y) = f_X(x) \times f_Y(y)
$$

If this factorization holds, then X and Y are independent.

In our case, since the joint density is:

$$
f_{X,Y}(x,y) = x e^x (1 + y)
$$

which is the product of $x e^x$ and $(1 + y)$ (up to constants), it suggests that the variables are independent if the densities are properly normalized.

Final conclusion for case 1:

- The joint density function factorizes into a product of functions of x and y.
- Therefore, X and Y are independent under assumptions of proper normalization and support.

Case 2: Joint Density $f(x,y) = 6 x y^2$

Understanding the Density Function

The second joint density is:

$$
f_{X,Y}(x,y) = 6 x y^2
$$

Typically, such a density is defined over a specific support where the integral over that support equals 1.

Suppose the support is:

$$
0 \leq x \leq 1, \quad 0 \leq y \leq 1
$$

which is common for beta-like distributions.

Step 1: Verify the validity of the density

Calculate the total probability:

$$
\int_0^1 \int_0^1 6 x y^2 dy dx
$$

First, integrate over y:

\[

\int_0^1 y^2 dy = \frac{1}{3}
\]

Then, over \(x\):

\[
\int_0^1 6 x \times \frac{1}{3} dx = 2 \int_0^1 x dx = 2 \times \frac{1}{2} =
1
\]

Thus, the density integrates to 1 over the support, confirming its validity.

Step 2: Find the marginal densities

- Marginal of X:

\[
f_X(x) = \int_0^1 6 x y^2 dy = 6 x \left[\frac{y^3}{3} \right]_0^1 = 6 x
\times \frac{1}{3} = 2 x
\]

- Marginal of Y:

\[
f_Y(y) = \int_0^1 6 x y^2 dx = 6 y^2 \left[\frac{x^2}{2} \right]_0^1 = 6 y^2
\times \frac{1}{2} = 3 y^2
\]

Step 3: Check for factorization

Express the joint density in terms of the marginals:

\[
f_{X,Y}(x,y) \stackrel{?}{=} f_X(x) \times f_Y(y)
\]

Calculate:

\[
f_X(x) \times f_Y(y) = (2 x) \times (3 y^2) = 6 x y^2
\

Suppose X And Y Have Joint Density F X Y Are X And Y

Independent Ifa F X Y Xe X 1 Y For X Y0 B F X Y 6xy 2

Suppose X And Y Have Joint Density F X Y Are X And Y Independent Ifa F X Y Xe X 1 Y For X Y0 B F X Y 6xy 2

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