Use Green's Theorem To Calculate The Circulation Of F= Yi+2xyj Around The Unit Circle, Oriented Counterclockwise.circulation

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Introduction to Green's Theorem and Circulation

Green's Theorem is a fundamental result in vector calculus that relates the line integral around a simple closed curve to a double integral over the region enclosed by the curve. It serves as a powerful tool for converting complex line integrals into more manageable double integrals, especially when dealing with circulation and flux problems.

Circulation, in the context of vector fields, refers to the line integral of a vector field along a closed curve. Specifically, it measures the tendency of the field to circulate around that curve. The problem at hand involves calculating the circulation of the vector field F = Yi + 2xyj around the unit circle, oriented counterclockwise, using Green's Theorem.

Understanding the Given Vector Field

Vector Field Components

The vector field F is given as:

$$
\mathbf{F} = Y \mathbf{i} + 2xy \mathbf{j}
$$

which can be expressed in component form as:

$$
\mathbf{F}(x, y) = (P(x, y), Q(x, y))
$$

where:
- $P(x, y) = y$
- $Q(x, y) = 2xy$

Understanding these components is essential for applying Green's Theorem, as it involves derivatives of P and Q.

Green's Theorem: Statement and Application

Statement of Green’s Theorem

Green's Theorem relates the circulation of a vector field around a simple, positively oriented, closed curve C to a double integral over the region D enclosed by C:

$$
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy
$$

where:
- $\mathbf{F} = (P, Q)$
- C is positively oriented (counterclockwise)
- D is the region enclosed by C

The left side of the equation is the line integral (circulation), and the right side is a double integral involving derivatives of P and Q.

Applying Green's Theorem to the Problem

To find the circulation of F around the unit circle, we need to evaluate:

$$
\oint_{C} \mathbf{F} \cdot d\mathbf{r}
$$

using Green's Theorem, which converts this into:

$$
\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy
$$

where D is the unit disk $x^2 + y^2 \leq 1$.

Calculating the Necessary Derivatives

Partial Derivatives of P and Q

Compute the derivatives:
- $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} (2xy) = 2y$
- $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} (y) = 1$

Thus, the integrand becomes:

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y - 1
$$

Setting Up the Double Integral

The circulation is now expressed as:

$$
\text{Circulation} = \iint_D (2y - 1) dxdy
$$

where D is the unit disk $x^2 + y^2 \leq 1$.

Since the region D is a circle, converting to polar coordinates simplifies the integration.

Polar Coordinates Transformation

Recall:

$$
x = r \cos \theta,\quad y = r \sin \theta,\quad dxdy = r dr d\theta
$$

and the limits:

$$
r: 0 \to 1,\quad \theta: 0 \to 2\pi
$$

Express the integrand in polar coordinates:

$$
2y - 1 = 2(r \sin \theta) - 1
$$

The double integral becomes:

$$
\int_{0}^{2\pi} \int_{0}^{1} [2r \sin \theta - 1] r dr d\theta
$$

Evaluating the Double Integral

Integral in Detail

Break the integral into two parts:

$$
\text{Circulation} = \int_{0}^{2\pi} \int_{0}^{1} 2r^2 \sin \theta dr d\theta - \int_{0}^{2\pi} \int_{0}^{1} r dr d\theta
$$

Evaluate each separately:

1. First integral:

$$
I_1 = \int_{0}^{2\pi} \left(\int_{0}^{1} 2r^2 \sin \theta dr \right) d\theta
$$

Since $\sin \theta$ does not depend on r:

$$
I_1 = \int_{0}^{2\pi} \sin \theta \left(\int_{0}^{1} 2r^2 dr \right) d\theta
$$

Calculate the inner integral:

$$
\int_{0}^{1} 2r^2 dr = 2 \times \frac{r^3}{3} \Big|_0^1 = \frac{2}{3}
$$

Thus:

$$
I_1 = \frac{2}{3} \int_{0}^{2\pi} \sin \theta d\theta
$$

And:

$$
\int_{0}^{2\pi} \sin \theta d\theta = 0
$$

Hence:

$$
I_1 = 0
$$

2. Second integral:

$$
I_2 = \int_{0}^{2\pi} \left(\int_{0}^{1} r dr \right) d\theta
$$

Inner integral:

$$
\int_{0}^{1} r dr = \frac{r^2}{2} \Big|_0^1 = \frac{1}{2}
$$

Therefore:

$$
I_2 = \frac{1}{2} \int_{0}^{2\pi} d\theta = \frac{1}{2} \times 2\pi = \pi
$$

Final Calculation and Result

Putting it all together:

$$
\text{Circulation} = I_1 - I_2 = 0 - \pi = -\pi
$$

Thus, the circulation of the vector field $\mathbf{F} = y \mathbf{i} + 2xy \mathbf{j}$ around the unit circle oriented counterclockwise is $-\pi$.

Conclusion and Significance

Using Green's Theorem simplifies the process of calculating circulation for complex vector fields over closed curves, converting the line integral into a more straightforward double integral. In this particular case, the symmetry of the circle and the properties of the integrand led to the cancellation of certain terms, resulting in a simple final value of $-\pi$.

Understanding how to apply Green's Theorem not only streamlines calculations but also deepens insight into the behavior of vector fields and their circulation properties. This approach is widely applicable in fluid dynamics, electromagnetism, and other branches of physics and engineering where circulation and flux are fundamental concepts.

Additional Tips for Applying Green's Theorem

- Always verify that the curve is positively oriented (counterclockwise).
- Confirm that the region enclosed is simple and doesn't intersect itself.
- Convert to polar coordinates when dealing with circular regions to simplify integrations.
- Carefully compute partial derivatives and integrals to avoid errors.

By mastering these techniques, students and professionals can efficiently analyze circulation and flux in various physical and mathematical contexts.

Frequently Asked Questions

What is Green's Theorem and how is it used to calculate circulation around a closed curve?

Green's Theorem relates the line integral around a closed curve to a double integral over the region it encloses. Specifically, it states that the circulation of a vector field F around a simple, closed curve C is equal to the double integral of the curl of F over the region D bounded by C, i.e., $\oint_C F \cdot dr = \iint_D (\partial Q/\partial x - \partial P/\partial y)\, dx\, dy$.

How do you identify the components of the vector field F= Yi + 2xyj for applying Green's Theorem?

In the vector field F= Yi + 2xyj, the component functions are $P(x,y)= y$ and $Q(x,y)= 2xy$, corresponding to the i and j directions, respectively. These are used in the curl calculation for Green's Theorem.

What is the curl of the vector field F= Yi + 2xyj, and how is it computed?

The curl $(\partial Q/\partial x - \partial P/\partial y)$ for F= Yi + 2xyj is computed as follows: $\partial Q/\partial x = \partial(2xy)/\partial x = 2y$, and $\partial P/\partial y= \partial(y)/\partial y= 1$. Therefore, curl = $2y - 1$.

How do you set up the double integral over the unit circle to find the circulation using Green's Theorem?

Since the curve is the unit circle $x^2 + y^2 = 1$, the region D is the unit disk. The double integral becomes $\iint_D (2y - 1)\, dx\, dy$, where D is the disk of radius 1 centered at the origin. It can be evaluated using polar coordinates for simplicity.

What is the process for converting the double integral over the unit circle into polar coordinates?

In polar coordinates, $x= r \cos \theta$, $y= r \sin \theta$, with r from 0 to 1 and θ from 0 to 2π. The differential area element dx dy becomes $r\, dr\, d\theta$. The integral becomes $\iint_D (2r \sin \theta - 1)\, r\, dr\, d\theta$ over r=0 to 1 and θ=0 to 2π.

How do you evaluate the integral of the function (2r sin θ - 1) r over the unit disk?

The integral splits into two parts: $\int_0^{2\pi} \int_0^1 (2r \sin \theta)\, r\, dr\, d\theta - \int_0^{2\pi}$

$\int_0^1 r\, dr\, d\theta$. The first part involves integrating $\sin \theta$ over 0 to 2π, which is zero, simplifying the calculation. The second part evaluates to a finite value, giving the total circulation.

What is the final value of the circulation around the unit circle for the given vector field?

The final value of the circulation is obtained by evaluating the integrals, resulting in a value of $-\pi$, indicating the net rotation of the vector field around the circle.

Why is it important to verify the orientation of the curve when applying Green's Theorem?

Green's Theorem assumes the curve is positively oriented (counterclockwise). If the orientation is reversed, the sign of the circulation integral changes. Ensuring the correct orientation guarantees accurate calculation of the circulation.

What are common mistakes to avoid when calculating circulation using Green's Theorem for this problem?

Common mistakes include incorrect identification of P and Q, forgetting the orientation of the curve, miscomputing the curl, neglecting to convert to polar coordinates properly, or integrating over the wrong region. Carefully verifying each step minimizes errors.

Additional Resources

Use Green's Theorem To Calculate The Circulation Of F= Yi+2xyj Around The Unit Circle, Oriented Counterclockwise

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Introduction

In the realm of vector calculus, understanding the behavior of vector fields and their associated circulation and flux is fundamental. Among the most powerful tools to evaluate line integrals around closed curves is Green's Theorem, which establishes a profound relationship between the circulation of a vector field along a simple, closed curve and the double integral of its curl over the region enclosed by the curve. This article provides a comprehensive exploration of how Green's Theorem can be applied to compute the circulation of a specific vector field, $F = yi + 2xyj$, around the unit circle oriented counterclockwise.

Understanding the Fundamentals

Green's Theorem: An Overview

Green's Theorem fundamentally connects the line integral of a vector field around a simple, closed, positively oriented curve C with a double integral over the region D enclosed by C:

$$
\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx\,dy
$$

where $\mathbf{F} = P(x,y) \mathbf{i} + Q(x,y) \mathbf{j}$.

This theorem allows us to convert a potentially complicated line integral into a simpler double integral, provided the vector field and the curve satisfy certain conditions (smoothness, the curve being simple and closed, etc.).

The Concept of Circulation

Circulation refers to the tendency of a vector field to "rotate" or "circulate" around a closed curve. Mathematically, the circulation of a vector field $\mathbf{F}$ around a curve C is given by:

$$
\text{Circulation} = \oint_{C} \mathbf{F} \cdot d\mathbf{r}
$$

In our case, the goal is to compute the circulation of the field $\mathbf{F} = yi + 2xyj$ around the unit circle $x^2 + y^2 = 1$, with positive (counterclockwise) orientation.

Step-by-Step Application of Green's Theorem

1. Identify the Components P and Q

Given the vector field:

$$
\mathbf{F} = yi + 2xyj
$$

we can write:

$$
P(x, y) = y, \quad Q(x, y) = 2xy
$$

These are the functions that define the field's components in the x and y directions.

2. Compute the Partial Derivatives

Green's Theorem involves the difference between the partial derivatives:

$$
\frac{\partial Q}{\partial x} \quad \text{and} \quad \frac{\partial P}{\partial y}
$$

Calculate each:

$$
\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} (2xy) = 2y
$$
$$
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} (y) = 1
$$

3. Set Up the Double Integral

Applying Green's Theorem:

$$
\text{Circulation} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_D (2y - 1) \, dx\,dy
$$

where D is the unit disk $x^2 + y^2 \leq 1$.

Computing the Double Integral over the Unit Disk

The key step is evaluating:

$$
\iint_{x^2 + y^2 \leq 1} (2y - 1) \, dx\,dy
$$

This integral can be simplified using polar coordinates, which are well-suited for circular regions.

1. Transformation to Polar Coordinates

Recall:

$$
x = r \cos \theta, \quad y = r \sin \theta

\]

$$
dx\,dy = r dr d\theta
$$

$$
r \in [0, 1], \quad \theta \in [0, 2\pi]
$$

Express the integrand $2y - 1$:

$$
2y - 1 = 2 r \sin \theta - 1
$$

2. Set up the Integral in Polar Coordinates

$$
\iint_D (2y - 1) dx dy = \int_{0}^{2\pi} \int_{0}^{1} (2 r \sin \theta - 1) r dr d\theta
$$

which simplifies to:

$$
\int_{0}^{2\pi} \int_{0}^{1} (2 r^2 \sin \theta - r) dr d\theta
$$

Evaluating the Integral

The double integral separates into two parts:

$$
\int_{0}^{2\pi} \left[\int_{0}^{1} 2 r^2 \sin \theta dr - \int_{0}^{1} r dr \right] d\theta
$$

but note that the integrand contains $\sin \theta$, which depends only on θ, while the integrals over r are straightforward.

1. Inner Integrals over r

- First term:

$$
\int_{0}^{1} 2 r^2 \sin \theta dr = 2 \sin \theta \int_0^1 r^2 dr = 2 \sin \theta \left[\frac{r^3}{3} \right]_0^1 = 2 \sin \theta \times \frac{1}{3} = \frac{2}{3} \sin \theta
$$

- Second term:

$$
\int_0^1 r dr = \left[\frac{r^2}{2} \right]_0^1 = \frac{1}{2}
$$

2. The Entire θ-Integral

Putting it together:

$$
\int_0^{2\pi} \left(\frac{2}{3} \sin \theta - \frac{1}{2} \right) d\theta
$$

which splits into:

$$
\frac{2}{3} \int_0^{2\pi} \sin \theta d\theta - \frac{1}{2} \int_0^{2\pi} d\theta
$$

Calculate each:

- $\int_0^{2\pi} \sin \theta d\theta = 0$, since the sine function is symmetric over the interval.
- $\int_0^{2\pi} d\theta = 2\pi$

Thus:

$$
\text{Circulation} = \frac{2}{3} \times 0 - \frac{1}{2} \times 2\pi = -\pi
$$

Final Result and Interpretation

The circulation of the vector field $\mathbf{F} = yi + 2xyj$ around the unit circle (oriented counterclockwise) is:

$$
\boxed{
\text{Circulation} = -\pi
}
$$

This negative value indicates that the overall circulation is clockwise, opposite to the positive (counterclockwise) orientation specified. However, since the calculation was performed assuming positive orientation, the negative result reflects the actual directionality of the vector field relative to the chosen orientation.

Significance of the Result

This calculation exemplifies the power of Green’s Theorem in simplifying complex line integrals into manageable double integrals, especially over symmetric regions like the unit disk. The result not only quantifies the tendency of the field to circulate around the circle but also underscores the influence of the field's components, particularly how the y-dependent term impacts the overall circulation.

The negative sign suggests that, on average, the field tends to rotate clockwise around the circle, which might seem counterintuitive given the initial positive orientation. This highlights the importance of careful interpretation of signs when applying Green’s Theorem.

Broader Implications and Applications

Physical Contexts

Calculating circulation using Green’s Theorem has broad applications in physics and engineering, including:

- Fluid dynamics: Analyzing vorticity and circulation in fluid flows.
- Electromagnetism: Understanding magnetic field circulation.
- Meteorology: Studying circulation patterns in atmospheric flows.

Mathematical Significance

This approach demonstrates how topological and analytical tools intersect, providing insights into the structure of vector fields. It also exemplifies how symmetry and coordinate transformations can simplify integrals over complex regions.

Conclusion

In conclusion, Green’s Theorem offers a robust method for evaluating the circulation of vector fields around closed curves. Applying it to the vector field $\mathbf{F} = yi + 2xyj$ around the unit circle reveals a circulation of $-\pi$, quantifying the field’s rotational tendency. This process underscores

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Around The Unit Circle Oriented Counterclockwise Circulation

Use Green S Theorem To Calculate The Circulation Of F Yi 2xyj Around The Unit Circle Oriented Counterclockwise Circulation

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